Tracts in Mathematics 7

Hans Triebel

Function Spaces and Wavelets on Domains

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Preface

This book may be considered as a continuation of the monographs [T83], [T92], [T06]. Now we are mainly interested in spaces on domains in $\mathbb{R}^n$, related wavelet bases and wavelet frames, and extension problems. But first we deal in Chapter 1 with the usual spaces on $\mathbb{R}^n$, periodic spaces on $\mathbb{R}^n$ and on the n-torus $\mathbb{T}^n$ and their wavelet expansions under natural restrictions for the parameters involved. Spaces on arbitrary domains are the subject of Chapter 2. The heart of the exposition are the Chapters 3, 4, where we develop a theory of function spaces on so-called thick domains, including wavelet expansions and extensions to corresponding spaces on $\mathbb{R}^n$. This will be complemented in Chapter 5 by spaces on smooth manifolds and smooth domains. Finally we add in Chapter 6 a discussion about desirable properties of wavelet expansions in function spaces introducing the notation of Riesz wavelet bases and frames. This chapter deals also with some related topics, in particular with spaces on cellular domains.

Although we rely mainly on [T06] we repeat basic notation and a few classical assertions in order to make this text to some extent independently understandable and usable. More precisely, we have two types of readers in mind:

researchers in the theory of function spaces who are interested in wavelets as new effective building blocks, and

scientists who wish to use wavelet bases in *classical function spaces* in diverse applications.

Here is a ***guide*** to where one finds basic definitions and key assertions adapted to the second type of readers:

1. Classical Sobolev spaces $W^k_p(\mathbb{R}^n)$, Sobolev spaces $H^s_p(\mathbb{R}^n)$, classical Besov spaces $B^s_{pq}(\mathbb{R}^n)$ and Hölder–Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n)$ on the Euclidean n-spaces $\mathbb{R}^n$: Definition 1.1, Remark 1.2, p. 2.

2. Wavelets in $\mathbb{R}^n$: Section 1.2.1, p. 13.

3. Wavelet bases for spaces on $\mathbb{R}^n$: Theorem 1.20, p. 15.

4. Spaces on the n-torus $\mathbb{T}^n$: Definition 1.27, Remark 1.28, p. 21.

5. Wavelet bases for spaces on $\mathbb{T}^n$: Theorem 1.37, p. 26.

6. Spaces on arbitrary domains Ω in $\mathbb{R}^n$: Definitions 2.1, 5.17, Remark 2.2, pp. 28, 29, 147.

7. u-wavelet systems in domains Ω in $\mathbb{R}^n$: Definitions 2.4, 6.3, pp. 32, 179.

8. u-Riesz bases and u-Riesz frames: Definition 6.5, Section 6.2.2, pp. 180, 202.

9. Wavelet bases in $L_2(\Omega)$ and $L_p(\Omega)$ in arbitrary domains Ω in $\mathbb{R}^n$: Theorems 2.33, 2.36, 2.44, pp. 49, 53, 59.

10. Classes of domains Ω in $\mathbb{R}^n$ and their relations: Definitions 3.1, 3.4, 5.40, 6.9, Proposition 3.8, pp. 70, 72, 75, 168, 182.

11. Wavelet bases in E-thick domains (covering bounded Lipschitz domains) Ω in $\mathbb{R}^n$: Theorems 3.13, 3.23, Corollary 3.25, pp. 80, 89, 91.

12. Spaces, frames and bases on manifolds: Definitions 5.1, 5.5, 5.40, Theorems 5.9, 5.37, pp. 133, 135, 136, 164, 168.

13. Frames and bases on domains: Definition 5.25, Theorems 5.27, 5.35, 5.38, 5.51, 6.7, 6.30, 6.32, 6.33, pp. 152, 153, 162, 166, 175, 181, 196, 197, 198.

Formulas are numbered within chapters. Furthermore in each chapter all definitions, theorems, propositions, corollaries and remarks are jointly and consecutively numbered. Chapter n is divided in sections $n.k$ and subsections $n.k.l$. But when quoted we refer simply to Section $n.k$ or Section $n.k.l$ instead of Section $n.k$ or Subsection $n.k.l$. If there is no danger of confusion (which is mostly the case) we write $A^s_{pq}, B^s_{pq}, F^s_{pq}, \dots, a^s_{pq} \dots$(spaces) instead of $A^s_{p,q}, B^s_{p,q}, F^s_{p,q}, \dots, a^s_{p,q} \dots$. Similarly for $a_{jm}, \lambda_{jm}, Q_{jm}$ (functions, numbers, cubes) instead of $a_{j,m}, \lambda_{j,m}, Q_{j,m}$ etc. References are ordered by names, not by labels, which roughly coincides, but may occasionally cause minor deviations. The numbers behind ► in the Bibliography mark the page(s) where the corresponding entry is quoted (with the exception of [T78]–[T06]).

It is a pleasure to acknowledge the great help I have received from David Edmunds (Brighton) who looked through the manuscript and offered many comments.

Jena, Summer 2008 Hans Triebel

Contents

Chapter 1
Spaces on $\mathbb{R}^n$ and $\mathbb{T}^n$

1.1 Definitions, atoms, and local means

1.1.1 Definitions

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the usual Schwartz space and $S'(\mathbb{R}^n)$ be the space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$, is the standard quasi-Banach space with respect to the Lebesgue measure in $\mathbb{R}^n$, quasi-normed by

$$\|f \,|L_p(\mathbb{R}^n)\| = \Big(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \Big)^{1/p} \tag{1.1}$$

with the natural modification if $p = \infty$. As usual, $\mathbb{Z}$ is the collection of all integers; and $\mathbb{Z}^n$ where $n \in \mathbb{N}$, denotes the lattice of all points $m = (m_1, \dots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Let $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, be the set of all multi-indices,

$$\alpha = (\alpha_1, \dots, \alpha_n) \quad \text{with } \alpha_j \in \mathbb{N}_0 \text{ and } |\alpha| = \textstyle\sum_{j=1}^n \alpha_j. \tag{1.2}$$

If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ then we put

$$x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \quad \text{(monomials)}. \tag{1.3}$$

If $\varphi \in S(\mathbb{R}^n)$ then

$$\hat{\varphi}(\xi) = (F\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\, \varphi(x)\, \mathrm{d}x, \quad \xi \in \mathbb{R}^n, \tag{1.4}$$

denotes the Fourier transform of φ. As usual, $F^{-1}\varphi$ and $\varphi^\vee$ stand for the inverse Fourier transform, given by the right-hand side of (1.4) with i in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbb{R}^n$. Both F and F^{-1} are extended to $S'(\mathbb{R}^n)$ in the standard way. Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \ \text{ if } \ |x| \le 1 \quad \text{and} \quad \varphi_0(y) = 0 \ \text{ if } \ |y| \ge 3/2, \tag{1.5}$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n,\ k \in \mathbb{N}. \tag{1.6}$$

Since

$$\sum_{j=0}^\infty \varphi_j(x) = 1 \quad \text{for } x \in \mathbb{R}^n, \tag{1.7}$$

the φ_j form a dyadic resolution of unity. The entire analytic functions $(\varphi_j \hat{f})^\vee(x)$ make sense pointwise in $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$.

Definition 1.1. Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be the above dyadic resolution of unity.

(i) Let

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}. \tag{1.8}$$

Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\|_\varphi = \Big(\sum_{j=0}^\infty 2^{jsq} \|(\varphi_j \hat{f})^\vee \,|L_p(\mathbb{R}^n)\|^q \Big)^{1/q} < \infty \tag{1.9}$$

(with the usual modification if $q = \infty$).

(ii) Let

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}. \tag{1.10}$$

Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\|_\varphi = \Big\| \Big(\sum_{j=0}^\infty 2^{jsq} \big|(\varphi_j \hat{f})^\vee(\cdot)\big|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty \tag{1.11}$$

(with the usual modifications if $q = \infty$).

Remark 1.2. The theory of these spaces may be found in [T83], [T92], [T06]. In particular these spaces are independent of admitted resolutions of unity φ according to (1.5)–(1.7) (equivalent quasi-norms). This justifies our omission of the subscript φ in (1.9), (1.11) in the sequel. We remind the reader of a few special cases and properties referring for details to the above books, especially to [T06], Section 1.2.

(i) Let $1 < p < \infty$. Then

$$L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n) \tag{1.12}$$

is a well-known *Paley–Littlewood theorem*.

(ii) Let $1 < p < \infty$ and $k \in \mathbb{N}_0$. Then

$$W^k_p(\mathbb{R}^n) = F^k_{p,2}(\mathbb{R}^n) \tag{1.13}$$

are the *classical Sobolev spaces* usually equivalently normed by

$$\|f \,|W^k_p(\mathbb{R}^n)\| = \Big(\sum_{|\alpha| \le k} \|D^\alpha f \,|L_p(\mathbb{R}^n)\|^p \Big)^{1/p}. \tag{1.14}$$

This generalises (1.12).

(iii) Let $\sigma \in \mathbb{R}$. Then

$$I_\sigma : f \mapsto \big(\langle \xi \rangle^\sigma \hat{f} \big)^\vee \tag{1.15}$$

with $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, is an one-to-one map of $S(\mathbb{R}^n)$ onto itself and of $S'(\mathbb{R}^n)$ onto itself. Then I_σ is a lift for the spaces $A^s_{pq}(\mathbb{R}^n)$ with $A = B$ or $A = F$ and $s \in \mathbb{R}$, $0 < p \le \infty$ ($p < \infty$ for the F-scale), $0 < q \le \infty$:

$$I_\sigma A^s_{pq}(\mathbb{R}^n) = A^{s-\sigma}_{pq}(\mathbb{R}^n) \tag{1.16}$$

(equivalent quasi-norms). With

$$H^s_p(\mathbb{R}^n) = I_{-s} L_p(\mathbb{R}^n), \quad s \in \mathbb{R},\ 1 < p < \infty, \tag{1.17}$$

one has

$$H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n), \quad s \in \mathbb{R},\ 1 < p < \infty, \tag{1.18}$$

and

$$H^k_p(\mathbb{R}^n) = W^k_p(\mathbb{R}^n), \quad 1 < p < \infty,\ k \in \mathbb{N}_0. \tag{1.19}$$

Nowadays one calls the $H^s_p(\mathbb{R}^n)$ *Sobolev spaces* (sometimes fractional Sobolev spaces or Bessel-potential spaces) with the classical Sobolev spaces $W^k_p(\mathbb{R}^n)$ as a special case.

(iv) We denote

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n), \quad s \in \mathbb{R}, \tag{1.20}$$

as *Hölder–Zygmund spaces*. Let

$$(\Delta^1_h f)(x) = f(x+h) - f(x), \quad (\Delta^{l+1}_h f)(x) = \Delta^1_h (\Delta^l_h f)(x), \tag{1.21}$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $l \in \mathbb{N}$, be the iterated differences in $\mathbb{R}^n$. Let $0 < s < m \in \mathbb{N}$. Then

$$\| f \,|\mathcal{C}^s(\mathbb{R}^n)\|_m = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup |h|^{-s} |\Delta^m_h f(x)|, \tag{1.22}$$

where the second supremum is taken over all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$ with $0 < |h| \le 1$, are equivalent norms in $\mathcal{C}^s(\mathbb{R}^n)$.

(v) This assertion can be generalised as follows. Once more let $0 < s < m \in \mathbb{N}$ and $1 \le p, q \le \infty$. Then

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\|_m = \| f \,|L_p(\mathbb{R}^n)\| + \left(\int_{|h|\le 1} |h|^{-sq} \, \|\Delta^m_h f \,|L_p(\mathbb{R}^n)\|^q \frac{\mathrm{d}h}{|h|^n} \right)^{1/q} \tag{1.23}$$

and

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\|^*_m = \| f \,|L_p(\mathbb{R}^n)\| + \left(\int_0^1 t^{-sq} \sup_{|h|\le t} \|\Delta^m_h f \,|L_p(\mathbb{R}^n)\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} \tag{1.24}$$

(with the usual modification if $q = \infty$) are equivalent norms in $B^s_{pq}(\mathbb{R}^n)$. These are the *classical Besov spaces*. We refer to [T92], Chapter 1, and [T06], Chapter 1, where one finds the history of these spaces, further special cases and classical assertions. In addition, (1.23), (1.24) remain to be equivalent quasi-norms in

$$B^s_{pq}(\mathbb{R}^n) \quad \text{with } 0 < p, q \le \infty \text{ and } s > n\big(\tfrac{1}{p} - 1\big)_+. \tag{1.25}$$

1.1.2 Atoms

We give a detailed description of atomic representations of the spaces introduced in Definition 1.1 for two reasons. First we need these assertions later on. Secondly, atoms and local means, which are the subject of Section 1.1.3, are dual to each other where the natural smoothness assumptions and the cancellation conditions change their roles.

Let Q_{jm} be cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-j}m$ with side-length 2^{-j+1} where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side-length r times of the side-length of Q. Let χ_{jm} be the characteristic function of Q_{jm}.

Definition 1.3. Let $0 < p \le \infty$, $0 < q \le \infty$. Then b_{pq} is the collection of all sequences

$$\lambda = \{\lambda_{jm} \in \mathbb{C} : \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n\} \tag{1.26}$$

such that

$$\|\lambda \,|b_{pq}\| = \Big(\sum_{j=0}^{\infty} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}|^p \Big)^{q/p}\Big)^{1/q} < \infty, \tag{1.27}$$

and f_{pq} is the collection of all sequences λ according to (1.26) such that

$$\|\lambda \,| f_{pq}\| = \Big\| \Big(\sum_{j,m} 2^{jnq/p} \,|\lambda_{jm}\, \chi_{jm}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n)\Big\| < \infty \tag{1.28}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

Remark 1.4. Note that the factor $2^{jnq/p}$ in (1.28) does not appear if one relies on the p-normalised characteristic function $\chi^{(p)}_{jm}(x) = 2^{n(j-1)/p} \chi_{jm}(x)$. This is the usual way to say what is meant by f_{pq}. But for what follows the above version seems to be more appropriate. Of course $b_{pp} = f_{pp}$.

Definition 1.5. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$ and $d \ge 1$. Then the L_∞-functions $a_{jm} : \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called (s,p)-atoms if

$$\operatorname{supp} a_{jm} \subset d\, Q_{jm}, \quad j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n; \tag{1.29}$$

there exist all (classical) derivatives $D^\alpha a_{jm}$ with $|\alpha| \le K$ such that

$$|D^\alpha a_{jm}(x)| \le 2^{-j(s-\frac{n}{p})+j|\alpha|}, \quad |\alpha| \le K, \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{1.30}$$

and

$$\int_{\mathbb{R}^n} x^\beta \, a_{jm}(x)\,\mathrm{d}x = 0, \quad |\beta| < L, \ j \in \mathbb{N}, \ m \in \mathbb{Z}^n. \tag{1.31}$$

Remark 1.6. No cancellation (1.31) for $a_{0,m}$ is required. Furthermore, if $L = 0$ then (1.31) is empty (no condition). If $K = 0$ then (1.30) means $a_{jm} \in L_\infty$ and $|a_{jm}(x)| \le 2^{-j(s-\frac{n}{p})}$. Of course, the conditions for the above atoms depend not only on s and p, but also on the given numbers K, L, d. But this will only be indicated when extra clarity is required. Otherwise we speak about (s, p)-atoms instead of $(s, p)_{K,L,d}$-atoms. Let as usual

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \quad \text{and} \quad \sigma_{pq} = n\left(\frac{1}{\min(p,q)} - 1\right)_+ \tag{1.32}$$

where $b_+ = \max(b, 0)$ if $b \in \mathbb{R}$.

Theorem 1.7. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, $d \in \mathbb{R}$ *with*

$$K > s, \quad L > \sigma_p - s, \tag{1.33}$$

and $d \ge 1$ *be fixed. Then* $f \in S'(\mathbb{R}^n)$ *belongs to* $B^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\, a_{jm} \tag{1.34}$$

where the a_{jm} *are* (s, p)*-atoms (more precisely* $(s, p)_{K,L,d}$*-atoms) according to Definition 1.5 and* $\lambda \in b_{pq}$. *Furthermore*

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|b_{pq}\| \tag{1.35}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.34) (for fixed K, L, d*).*

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, $d \in \mathbb{R}$ *with*

$$K > s, \quad L > \sigma_{pq} - s, \tag{1.36}$$

and $d \ge 1$ *be fixed. Then* $f \in S'(\mathbb{R}^n)$ *belongs to* $F^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented by (1.34) where* a_{jm} *are* (s, p)*-atoms (more precisely* $(s, p)_{K,L,d}$*-atoms) according to Definition 1.5 and* $\lambda \in f_{pq}$. *Furthermore,*

$$\| f \,|F^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|f_{pq}\| \tag{1.37}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.34) (for fixed K, L, d*).*

Remark 1.8. Recall that dQ_{jm} are cubes centred at $2^{-j}m$ with side-length $d\, 2^{-j+1}$ where $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. For fixed d with $d \ge 1$ and $j \in \mathbb{N}_0$ there is some overlap of the cubes dQ_{jm} where $m \in \mathbb{Z}^n$. This makes clear that Theorem 1.7 based on Definition 1.5 is reasonable. Otherwise the above formulations coincide essentially with [T06], Section 1.5.1. There one finds also some technical comments how the convergence in (1.34) must be understood. Atoms of the above type go back essentially to [FrJ85], [FrJ90]. But more details about the complex history of atoms may be found in [T92], Section 1.9.

1.1.3 Local means

We wish to derive estimates for local means which are dual to atomic representations according to Theorem 1.7 as far as smoothness assumptions and cancellation properties are concerned. First we collect some definitions. Let Q_{jm} be the same cubes as in the previous Section 1.1.2.

Definition 1.9. Let $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ and $C > 0$. Then the L_∞-functions $k_{jm} : \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called kernels (of local means) if

$$\operatorname{supp} k_{jm} \subset C Q_{jm}, \quad j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n; \tag{1.38}$$

there exist all (classical) derivatives $D^\alpha k_{jm}$ with $|\alpha| \le A$ such that

$$|D^\alpha k_{jm}(x)| \le 2^{jn+j|\alpha|}, \quad |\alpha| \le A, \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{1.39}$$

and

$$\int_{\mathbb{R}^n} x^\beta \, k_{jm}(x) \, \mathrm{d}x = 0, \quad |\beta| < B, \ j \in \mathbb{N}, \ m \in \mathbb{Z}^n. \tag{1.40}$$

Remark 1.10. No cancellation (1.40) for $k_{0,m}$ is required. Furthermore if $B = 0$ then (1.40) is empty (no condition). If $A = 0$ then (1.39) means $k_{jm} \in L_\infty$ and $|k_{jm}(x)| \le 2^{jn}$. Compared with Definition 1.5 for atoms we have different normalisations in (1.30) and (1.39) (also due to the history of atoms). We adapt the sequence spaces introduced in Definition 1.3 in connection with atoms to the above kernels.

Definition 1.11. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then $\bar{b}^s_{pq}$ is the collection of all sequences λ according to (1.26) such that

$$\|\lambda \,|\bar{b}^s_{pq}\| = \Big(\sum_{j=0}^\infty 2^{j(s-\frac{n}{p})q} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}|^p \Big)^{q/p} \Big)^{1/q} < \infty, \tag{1.41}$$

and $\bar{f}^s_{pq}$ is the collection of all sequences λ according to (1.26) such that

$$\|\lambda \,|\bar{f}^s_{pq}\| = \Big\| \Big(\sum_{j,m} 2^{jsq} \, |\lambda_{jm} \, \chi_{jm}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty \tag{1.42}$$

with the usual modification if $p = \infty$ and/or $q = \infty$.

Remark 1.12. The notation b^s_{pq} and f^s_{pq} (without bar) will be reserved for a slight modification of the above sequence spaces in connection with wavelet representations. One has $\bar{b}^s_{pp} = \bar{f}^s_{pp}$.

Definition 1.13. Let $f \in B^s_{pq}(\mathbb{R}^n)$ where $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Let k_{jm} be the kernels according to Definition 1.9 with $A > \sigma_p - s$ where σ_p is given by (1.32) and $B \in \mathbb{N}_0$. Then

$$k_{jm}(f) = (f, k_{jm}) = \int_{\mathbb{R}^n} k_{jm}(y) \, f(y) \, \mathrm{d}y, \quad j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{1.43}$$

are local means, considered as a dual pairing within $(S(\mathbb{R}^n), S'(\mathbb{R}^n))$. Furthermore,

$$k(f) = \{k_{jm}(f) : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}. \tag{1.44}$$

Remark 1.14. We justify the dual pairing (1.43). According to [T83], Theorems 2.11.2, 2.11.3, one has for the dual spaces of $B^s_{pp}(\mathbb{R}^n)$ that

$$B^s_{pp}(\mathbb{R}^n)' = B^{-s+\sigma_p}_{p'p'}(\mathbb{R}^n), \quad x \in \mathbb{R},\ 0 < p < \infty, \tag{1.45}$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{if } 1 \le p \le \infty \text{ and } p' = \infty \text{ if } 0 < p < 1. \tag{1.46}$$

Since $k_{jm} \in C^A(\mathbb{R}^n)$ ($L_\infty(\mathbb{R}^n)$ if $A = 0$) has compact support one obtains that $k_{jm} \in B^{A-\varepsilon}_{uv}(\mathbb{R}^n)$ for any $\varepsilon > 0$ and $0 < u, v \le \infty$. By

$$B^s_{pq}(\mathbb{R}^n) \subset B^{s-\varepsilon}_{pp}(\mathbb{R}^n) \quad \text{and} \quad B^s_{\infty q}(\mathbb{R}^n) \subset B^{s,\mathrm{loc}}_{pq}(\mathbb{R}^n) \tag{1.47}$$

locally for any $s \in \mathbb{R}$, $\varepsilon > 0$, $0 < p < \infty$ and $0 < q \le \infty$ one has by (1.45) and $A > \sigma_p - s$ that (1.43) makes always sense as a dual pairing. This applies also to $f \in F^s_{pq}(\mathbb{R}^n)$ since

$$F^s_{pq}(\mathbb{R}^n) \subset B^s_{p,\max(p,q)}(\mathbb{R}^n). \tag{1.48}$$

But (1.43) can also be justified for $f \in B^s_{pq}(\mathbb{R}^n)$ and $f \in F^s_{pq}(\mathbb{R}^n)$ as in [T06], Section 5.1.7, by direct arguments.

In Section 1.2 we characterise the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ in terms of wavelets. Since wavelets are special atoms one has by Theorem 1.7 the desired estimates from above. But wavelets can also be considered as kernels k_{jm} of corresponding local means. This gives finally the needed estimates from below which will be reduced to the following theorem which might be considered as the main assertion of Section 1.1.

Theorem 1.15. (i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Let k_{jm} be kernels according to Definition 1.9 where $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ with*

$$A > \sigma_p - s, \quad B > s, \tag{1.49}$$

and $C > 0$ are fixed. Let $k(f)$ be as in (1.43), (1.44). Then for some $c > 0$ and all $f \in B^s_{pq}(\mathbb{R}^n)$,

$$\|k(f) \,|\bar{b}^s_{pq}\| \le c\, \|f \,|B^s_{pq}(\mathbb{R}^n)\|. \tag{1.50}$$

(ii) *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Let k_{jm} and $k(f)$ be the above kernels where $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ with*

$$A > \sigma_{pq} - s, \quad B > s, \tag{1.51}$$

and $C > 0$ are fixed. Then for some $c > 0$ and all $f \in F^s_{pq}(\mathbb{R}^n)$,

$$\|k(f) \,|\bar{f}^s_{pq}\| \le c\, \|f \,|F^s_{pq}(\mathbb{R}^n)\|. \tag{1.52}$$

Proof. Step 1. We prove (i). By Remark 1.14 the local means $k_{jm}(f)$ with $f \in B^s_{pq}(\mathbb{R}^n)$ make sense. Let

$$f = \sum_{r=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \lambda_{rl}\, a_{rl}(x), \quad f \in B^s_{pq}(\mathbb{R}^n), \tag{1.53}$$

be an optimal atomic decomposition according to Theorem 1.7 with a_{rl} as in (1.29)–(1.31) where

$$K = B > s \quad \text{and} \quad L = A > \sigma_p - s. \tag{1.54}$$

We recall that an atomic decomposition is called *optimal* if for some suitably chosen fixed $D > 0$ and all $f \in B^s_{pq}(\mathbb{R}^n)$,

$$\|\lambda \,|b_{pq}\| \le D\, \|f \,|B^s_{pq}(\mathbb{R}^n)\|. \tag{1.55}$$

For $j \in \mathbb{N}_0$ we split (1.53) into

$$f = f_j + f^j = \sum_{r=0}^{j} \cdots + \sum_{r=j+1}^{\infty} \cdots . \tag{1.56}$$

Then

$$\int_{\mathbb{R}^n} k_{jm}(y)\, f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} k_{jm}(y)\, f_j(y)\,\mathrm{d}y + \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\,\mathrm{d}y. \tag{1.57}$$

Let $r \le j$ and let $l \in l^j_r(m)$ where C and d in

$$l^j_r(m) = \big\{ l \in \mathbb{Z}^n : \, CQ_{jm} \cap d\, Q_{rl} \neq \emptyset \big\} \tag{1.58}$$

have the same meaning as in (1.38), (1.29). Then card $l^j_r(m) \sim 1$ and

$$\begin{aligned} &2^{j(s-\frac{n}{p})} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, a_{rl}(y)\,\mathrm{d}y \right| \\ &\quad \le c\, 2^{j(s-\frac{n}{p})} \sum_{|\gamma|=B} \sup_x |D^\gamma a_{rl}(x)| \int_{\mathbb{R}^n} |k_{jm}(y)| \cdot |y - 2^{-j}m|^B\,\mathrm{d}y \\ &\quad \le c'\, 2^{(j-r)(s-\frac{n}{p})}\, 2^{rB}\, 2^{-jB}\, 2^{jn}\, 2^{-jn} \\ &\quad = c'\, 2^{(j-r)(s-\frac{n}{p}-B)}. \end{aligned} \tag{1.59}$$

Hence one obtains for any fixed $\varepsilon > 0$ that (modification if $p = \infty$)

$$2^{j(s-\frac{n}{p})p} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f_j(y)\,\mathrm{d}y \right|^p \le c \sum_{r=0}^{j} \sum_{l \in l^j_r(m)} |\lambda_{rl}|^p\, 2^{(j-r)(s-\frac{n}{p}-B+\varepsilon)p}. \tag{1.60}$$

For fixed r, j with $r \le j$ and $l \in \mathbb{Z}^n$ one has

$$\text{card}\, \big\{ m \in \mathbb{Z}^n : \, l \in l^j_r(m) \big\} \sim 2^{(j-r)n}. \tag{1.61}$$

Then it follows from (1.60) that

$$
\begin{aligned}
&2^{j(s-\frac{n}{p})p} \sum_{m\in\mathbb{Z}^n} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f_j(y)\, \mathrm{d}y \right|^p \\
&\le c \sum_{r=0}^{j} 2^{(j-r)n}\, 2^{p(j-r)(s-\frac{n}{p}-B+\varepsilon)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^p \qquad (1.62)\\
&= c \sum_{r=0}^{j} 2^{p(j-r)(s-B+\varepsilon)} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^p .
\end{aligned}
$$

Let $r > j$. Then one obtains by (1.54) for $l \in l_r^j(m)$ that

$$
\begin{aligned}
&2^{j(s-\frac{n}{p})} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, a_{rl}(y)\, \mathrm{d}y \right| \\
&\le c\, 2^{j(s-\frac{n}{p})} \sum_{|\gamma|=A} \sup_x |D^\gamma k_{jm}(x)| \int_{\mathbb{R}^n} |a_{rl}(y)| \cdot |y - 2^{-r} l|^A \, \mathrm{d}y \qquad (1.63)\\
&\le c'\, 2^{(j-r)(s-\frac{n}{p})}\, 2^{jn+Aj}\, 2^{-rA-rn} \\
&= c'\, 2^{(j-r)(s-\frac{n}{p}+n+A)} .
\end{aligned}
$$

By (1.56), (1.57) it follows that for any fixed $\varepsilon > 0$,

$$
2^{j(s-\frac{n}{p})} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\, \mathrm{d}y \right|^p \le c \sum_{r>j} 2^{(j-r)(s-\frac{n}{p}+n+A-\varepsilon)p} \Big(\sum_{l\in l_r^j(m)} |\lambda_{rl}| \Big)^p . \tag{1.64}
$$

If $p \le 1$ then one obtains

$$
2^{j(s-\frac{n}{p})p} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\, \mathrm{d}y \right|^p \le c \sum_{r>j} 2^{(j-r)(s-\frac{n}{p}+n+A-\varepsilon)p} \sum_{l\in l_r^j(m)} |\lambda_{rl}|^p . \tag{1.65}
$$

If $1 < p < \infty$ (modification if $p = \infty$) then it follows from (1.64), Hölder's inequality and card $l_r^j(m) \sim 2^{n(r-j)}$ that

$$
\begin{aligned}
&2^{j(s-\frac{n}{p})p} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\, \mathrm{d}y \right|^p \\
&\le c \sum_{r>j} 2^{(j-r)(s-\frac{n}{p}+n+A-\varepsilon)p} \sum_{l\in l_r^j(m)} 2^{(r-j)(p-1)n} |\lambda_{rl}|^p \qquad (1.66)\\
&= c \sum_{r>j} 2^{(j-r)(s+A-\varepsilon)p} \sum_{l\in l_r^j(m)} |\lambda_{rl}|^p .
\end{aligned}
$$

Summation over $m \in \mathbb{Z}^n$ results for all p with $0 < p \le \infty$ (modification if $p = \infty$)

in

$$2^{j(s-\frac{n}{p})p} \sum_{m\in\mathbb{Z}^n} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\, \mathrm{d}y \right|^p \le c \sum_{r>j} 2^{(j-r)(s-\sigma_p+A-\varepsilon)p} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^p. \tag{1.67}$$

Now one obtains by (1.62), (1.67) and (1.49) for some $\varkappa > 0$,

$$2^{j(s-\frac{n}{p})p} \sum_{m\in\mathbb{Z}^n} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f(y)\, \mathrm{d}y \right|^p \le c \sum_{r=0}^{\infty} 2^{-(j-r)\varkappa p} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^p. \tag{1.68}$$

Using (1.43), (1.41), (1.55) based on (1.27) we obtain by standard arguments that

$$\|k(f)\,|\bar{b}^s_{pq}\| \le c\, \|\lambda\,|b_{pq}\| \le c\, D\, \|f\,|B^s_{pq}(\mathbb{R}^n)\|. \tag{1.69}$$

This proves (1.50)

Step 2. The proof of part (ii) of the theorem will be based on the vector-valued maximal inequality

$$\left\| \Big(\sum_{k=0}^{\infty} \Big(M|g_k|^w \Big)(\cdot)^{q/w} \Big)^{1/q} |L_p(\mathbb{R}^n) \right\| \le c \left\| \Big(\sum_{k=0}^{\infty} |g_k(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{1.70}$$

due to [FeS71] where

$$0 < p < \infty, \quad 0 < q \le \infty, \quad 0 < w < \min(p,q). \tag{1.71}$$

Here M is the Hardy–Littlewood maximal function,

$$(Mg)(x) = \sup |Q|^{-1} \int_Q |g(y)|\, \mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{1.72}$$

where the supremum is taken over all cubes centred at x. A short proof may also be found in [Tor86], pp. 303–305. By Remark 1.14 the local means $k_{jm}(f)$ with $f \in F^s_{pq}(\mathbb{R}^n)$ make sense. Let

$$f = \sum_{r=0}^{\infty} \sum_{l\in\mathbb{Z}^n} \lambda_{rl}\, a_{rl}(x), \quad f \in F^s_{pq}(\mathbb{R}^n), \tag{1.73}$$

be an optimal atomic decomposition according to Theorem 1.7 with a_{rl} as in (1.29)–(1.31) where

$$K = B > s \quad \text{and} \quad L = A > \sigma_{pq} - s. \tag{1.74}$$

An atomic decomposition is called *optimal* if for some suitably chosen fixed $D > 0$ and all $f \in F^s_{pq}(\mathbb{R}^n)$,

$$\|\lambda\,|f_{pq}\| \le D\, \|f\,|F^s_{pq}(\mathbb{R}^n)\|. \tag{1.75}$$

We rely again on the splitting (1.56), (1.57). Let $r \le j$. We assume $q < \infty$ (if $q = \infty$ one has to modify what follows appropriately). Then the counterpart of (1.59), (1.60) is now given by

$$
\begin{aligned}
&2^{jsq}\, \chi_{jm}(x) \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f_j(y)\, \mathrm{d}y \right|^q \\
&\quad \le c \sum_{r=0}^{j} \sum_{l \in l_r^j(m)} |\lambda_{rl}|^q\, 2^{\frac{rn}{p}q}\, \chi_{jm}(x)\, 2^{(j-r)(s-B+\varepsilon)q}
\end{aligned}
\tag{1.76}
$$

where $x \in \mathbb{R}^n$. For fixed j, r, and l the summation $\sum \chi_{jm}(x)$ over those $m \in \mathbb{Z}^n$ with $l \in l_r^j(m)$ is comparable with $\chi_{rl}(x)$ and can be estimated from above by its maximal function. Hence we obtain for any $w > 0$,

$$
\begin{aligned}
&2^{jsq} \sum_{m \in \mathbb{Z}^n} \chi_{jm}(x) \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f_j(y)\, \mathrm{d}y \right|^q \\
&\quad \le c \sum_{r=0}^{j} 2^{(j-r)(s-B+\varepsilon)q} \sum_{l \in \mathbb{Z}^n} M\big(|\lambda_{rl}|^w\, 2^{\frac{rn}{p}w}\, \chi_{rl}(\cdot)\big)(x)^{q/w}.
\end{aligned}
\tag{1.77}
$$

This is the counterpart of (1.62). Let now $r > j$. Then it follows as in (1.63), (1.64) that

$$
2^{js}\, \chi_{jm}(x) \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\, \mathrm{d}y \right| \le c\, \chi_{jm}(x) \sum_{r>j} 2^{(j-r)(s+A+n)} \sum_{l \in l_r^j(m)} 2^{r\frac{n}{p}}\, |\lambda_{rl}|.
\tag{1.78}
$$

For $x \in \mathbb{R}^n$ with $\chi_{jm}(x) = 1$ and $0 < w < 1$ the last factor can be estimated by

$$
\begin{aligned}
\Big(\sum_{l \in l_r^j(m)} 2^{r\frac{n}{p}} |\lambda_{rl}| \Big)^w &\le \sum_{l \in l_r^j(m)} 2^{r\frac{nw}{p}}\, |\lambda_{rl}|^w \\
&\le c\, 2^{rn}\, 2^{-jn}\, 2^{jn} \int_{\mathbb{R}^n} \sum_{l \in l_r^j(m)} 2^{r\frac{nw}{p}}\, |\lambda_{rl}|^w\, \chi_{rl}(y)\, \mathrm{d}y \\
&\le c'\, 2^{(r-j)n}\, M\Big(\sum_{l \in l_r^j(m)} 2^{r\frac{nw}{p}}\, |\lambda_{rl}|^w\, \chi_{rl}(\cdot)\Big)(x).
\end{aligned}
\tag{1.79}
$$

Assuming again $q < \infty$ one obtains for any fixed $\varepsilon > 0$ that

$$
\begin{aligned}
&2^{jsq}\, \chi_{jm}(x) \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\, \mathrm{d}y \right|^q \\
&\quad \le c \sum_{r>j} 2^{(j-r)(s+A+n-\frac{n}{w}-\varepsilon)q} M\Big(\sum_{l \in l_r^j(m)} 2^{r\frac{nw}{p}} |\lambda_{rl}|^w\, \chi_{rl}(\cdot)\Big)(x)^{q/w}.
\end{aligned}
\tag{1.80}
$$

Since

$$A + s > \sigma_{pq} = \frac{n}{\min(1, p, q)} - n \tag{1.81}$$

one may choose w with $w < \min(1, p, q)$ and $\varepsilon > 0$ in (1.80) such that

$$\varkappa = A + s + n - \frac{n}{w} - \varepsilon > 0. \tag{1.82}$$

With $r = j + t$ we have

$$\begin{aligned} &2^{jsq} \sum_{m \in \mathbb{Z}^n} \chi_{jm}(x) \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\, \mathrm{d}y \right|^q \\ &\le c \sum_{t=1}^{\infty} 2^{-t\varkappa q} \sum_{m \in \mathbb{Z}^n} M\Big(\sum_{l \in l^j_{j+t}(m)} 2^{(j+t)\frac{nw}{p}} |\lambda_{j+t,l}|^w \chi_{j+t,l}(\cdot)\Big)(x)^{q/w}. \end{aligned} \tag{1.83}$$

Summation over j in (1.77) with $s - B + \varepsilon < 0$ and in (1.83) gives by (1.42) that

$$\begin{aligned} \|k(f)\,|\bar{f}^s_{pq}\| \le c\, &\Big\| \Big(\sum_{r=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \big[M\big(|\lambda_{rl}|^w\, 2^{\frac{rn}{p}w}\, \chi_{rl}(\cdot)\big)\big]^{q/w}\Big)^{1/q} |L_p(\mathbb{R}^n)\Big\| \\ &+ c \sum_{t=1}^{\infty} 2^{-t\varkappa'} \Big\| \Big(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \Big[M\Big(\sum_{l \in l^j_{j+t}(m)} \ldots \Big)\Big]^{q/w}\Big)^{1/q} |L_p(\mathbb{R}^n)\Big\| \end{aligned} \tag{1.84}$$

for some $0 < \varkappa' < \varkappa$. By construction, the lattice $2^{-j}\mathbb{Z}^n$ refined by $l \in l^j_{j+t}(m)$ is related to $2^{-j-t}\mathbb{Z}^n$. Since $w < \min(1, p, q)$ one can apply (1.70). Then one obtains by (1.28) and (1.75) that

$$\|k(f)\,|\bar{f}^s_{pq}\| \le c\, \|\lambda\,|f_{pq}\| \le c\, D\, \|f\,|F^s_{pq}(\mathbb{R}^n)\|. \tag{1.85}$$

This proves (1.52). □

Remark 1.16. We refer in this context also to [Kyr03]. In particular we took over from this paper the idea to decompose the lattice $2^{-j-t}\mathbb{Z}^n$ in connection with (1.83), (1.84) into clusters around $2^{-j}\mathbb{Z}^n$. These clusters disappear after the vector-valued maximal inequality has been applied. According to Definition 1.9 the kernels k_{jm} have compact supports. This was of some use for us. But there is little doubt that one can replace the compactness assumption (1.38) by a sufficiently strong decay. We refer again to [Kyr03].

Remark 1.17. We followed closely [Tri07c] (as always throughout Chapter 1). In particular, Theorem 1.15 coincides essentially with [Tri07c], Theorem 15, and its proof. But the first formulation of Theorem 1.15 was given in [Tri07b], Theorem 36 (without proof), where we used it in connection with para-wavelet bases in domains and sampling numbers.

1.2 Spaces on $\mathbb{R}^n$

1.2.1 Wavelets in $L_2(\mathbb{R}^n)$

Recall that this book might be considered as the continuation of [T06]. There we dealt in Section 3.1, with wavelet bases (isomorphisms) in the spaces

$$B^s_{pq}(\mathbb{R}^n), \quad F^s_{pq}(\mathbb{R}^n) \quad \text{where } s \in \mathbb{R},\ 0 < p, q \le \infty, \tag{1.86}$$

(with $p < \infty$ for the F-spaces) based on a weaker version of Theorem 1.15. We wish to improve these assertions using now Theorem 1.15. But first we fix some notation and recall a few well-known facts.

We suppose that the reader is familiar with wavelets in $\mathbb{R}^n$ of Daubechies type and the related multiresolution analysis. The standard references are [Dau92], [Mal99], [Mey92], [Woj97]. A short summary of what is needed may also be found in [T06], Section 1.7. We give a brief description of some basic notation. As usual, $C^u(\mathbb{R})$ with $u \in \mathbb{N}$ collects all (complex-valued) continuous functions on $\mathbb{R}$ having continuous bounded derivatives up to order u (inclusively). Let

$$\psi_F \in C^u(\mathbb{R}), \quad \psi_M \in C^u(\mathbb{R}), \quad u \in \mathbb{N}, \tag{1.87}$$

be *real* compactly supported Daubechies wavelets with

$$\int_{\mathbb{R}} \psi_M(x)\, x^v \,\mathrm{d}x = 0 \quad \text{for all } v \in \mathbb{N}_0 \text{ with } v < u. \tag{1.88}$$

Recall that ψ_F is called the *scaling function* (father wavelet) and ψ_M the associated *wavelet* (mother wavelet). We extend these wavelets from $\mathbb{R}$ to $\mathbb{R}^n$ by the usual tensor procedure. Let

$$G = (G_1, \dots, G_n) \in G^0 = \{F, M\}^n, \tag{1.89}$$

which means that G_r is either F or M. Let

$$G = (G_1, \dots, G_n) \in G^j = \{F, M\}^{n*}, \quad j \in \mathbb{N}, \tag{1.90}$$

which means that G_r is either F or M where $*$ indicates that at least one of the components of G must be an M. Hence G^0 has 2^n elements, whereas G^j with $j \in \mathbb{N}$ has $2^n - 1$ elements. Let

$$\Psi^j_{G,m}(x) = 2^{jn/2} \prod_{r=1}^n \psi_{G_r}(2^j x_r - m_r), \quad G \in G^j,\ m \in \mathbb{Z}^n, \tag{1.91}$$

where $j \in \mathbb{N}_0$. We always assume the ψ_F and ψ_M in (1.87) have L_2-norm 1. *Then*

$$\big\{ \Psi^j_{G,m} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n \big\} \tag{1.92}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$ (*for any* $u \in \mathbb{N}$) and

$$f = \sum_{j=0}^\infty \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G}_m \, 2^{-jn/2}\, \Psi^j_{G,m} \tag{1.93}$$

with

$$\begin{aligned}\lambda_m^{j,G} = \lambda_m^{j,G}(f) &= 2^{jn/2} \int_{\mathbb{R}^n} f(x)\, \Psi_{G,m}^j(x)\, \mathrm{d}x \\ &= 2^{jn/2} \big(f, \Psi_{G,m}^j\big)\end{aligned} \tag{1.94}$$

in the corresponding expansion, adapted to our needs, where $2^{-jn/2}\Psi_{G,m}^j$ are uniformly bounded functions (with respect to j and m).

1.2.2 Wavelets in $A^s_{pq}(\mathbb{R}^n)$

One may ask whether the orthonormal basis (1.92) in $L_2(\mathbb{R}^n)$ remains an (unconditional) basis in other spaces on $\mathbb{R}^n$. First candidates are $L_p(\mathbb{R}^n)$ with $1 < p < \infty$ but also related (fractional) Sobolev spaces and classical Besov spaces. Something may be found in the above-mentioned books [Dau92], [Mal99], [Mey92], [Woj97]. One may also consult [T06], Remarks 1.63, 1.65, pp. 32, 34, for more details and further references. An extension of this theory to all spaces

$$A^s_{pq}(\mathbb{R}^n) \quad \text{where either } A = B \text{ or } A = F \text{ and } s \in \mathbb{R},\, 0 < p, q \le \infty, \tag{1.95}$$

with $p < \infty$ for the F-spaces may be found in [Kyr03], [Tri04] and [T06], Section 3.1.3, Theorem 3.5, p. 154. Basically one identifies the wavelets in (1.91) with atoms and with kernels of local means as described in Theorems 1.7 and 1.15. But the sharp version of Theorem 1.15 was not available at this time. We had to rely on a weaker version (from the point of view of applications to wavelets) and obtained wavelet bases and isomorphisms for all spaces in (1.95) under somewhat unnatural restrictions for the required smoothness $u \in \mathbb{N}$ in (1.87) of the underlying wavelets.

Now we return to this topic and adapt first the sequence spaces in Definition 1.11 incorporating the additional parameter $G \in G^j$ according to (1.89), (1.90).

Definition 1.18. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then b^s_{pq} is the collection of all sequences

$$\lambda = \big\{\lambda_m^{j,G} \in \mathbb{C} :\ j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\big\} \tag{1.96}$$

such that

$$\|\lambda \,|b^s_{pq}\| = \bigg(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G\in G^j} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_m^{j,G}|^p\Big)^{q/p}\bigg)^{1/q} < \infty \tag{1.97}$$

and f^s_{pq} is the collection of all sequences (1.96) such that

$$\|\lambda \,|f^s_{pq}\| = \bigg\|\Big(\sum_{j,G,m} 2^{jsq}\, |\lambda_m^{j,G}\, \chi_{jm}(\cdot)|^q\Big)^{1/q} |L_p(\mathbb{R}^n)\bigg\| < \infty \tag{1.98}$$

with the usual modification if $p = \infty$ and/or $q = \infty$.

Remark 1.19. One may consult Remark 1.12 as far as notation is concerned. Recall that χ_{jm} is the characteristic function of the cube Q_{jm} as introduced in Section 1.1.2 and used in Definitions 1.3 and 1.11.

As justified at the beginning of [T06], Section 3.1.3, we may abbreviate the right-hand side of (1.93) by

$$\sum_{j,G,m} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi^j_{G,m} \tag{1.99}$$

since the conditions for the sequences λ always ensure that the corresponding series converges unconditionally at least in $S'(\mathbb{R}^n)$ (which means that any rearrangement converges in $S'(\mathbb{R}^n)$ and has the same limit). *Local convergence* in $B^\sigma_{pq}(\mathbb{R}^n)$ means convergence in $B^\sigma_{pq}(K)$ for any ball K in $\mathbb{R}^n$. Similarly for $F^\sigma_{pq}(\mathbb{R}^n)$. Recall that σ_p and σ_{pq} are given by (1.32).

Otherwise we use standard notation naturally extended from Banach spaces to quasi-Banach spaces. In particular $\{b_j\}_{j=1}^\infty \subset B$ in a separable complex quasi-Banach B space is called a *basis* if any $b \in B$ can be uniquely represented as

$$b = \sum_{j=1}^\infty \lambda_j\, b_j, \quad \lambda_j \in \mathbb{C} \quad \text{(convergence in } B\text{)}. \tag{1.100}$$

A basis $\{b_j\}_{j=1}^\infty$ is called an *unconditional basis* if for any rearrangement σ of $\mathbb{N}$ (one-to-one map of $\mathbb{N}$ onto itself) $\{b_{\sigma(j)}\}_{j=1}^\infty$ is again a basis and

$$b = \sum_{j=1}^\infty \lambda_{\sigma(j)}\, b_{\sigma(j)} \quad \text{(convergence in } B\text{)} \tag{1.101}$$

for any $b \in B$ with (1.100). Standard bases of (separable) sequence spaces as considered in this book are always unconditional. We refer to [AlK06] for details about bases in Banach (sequence) spaces.

Theorem 1.20. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ *and let* $\Psi^j_{G,m}$ *be the wavelets* (1.91) *based on* (1.87), (1.88) *with*

$$u > \max(s, \sigma_p - s). \tag{1.102}$$

Let $f \in S'(\mathbb{R}^n)$. *Then* $f \in B^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi^j_{G,m}, \quad \lambda \in b^s_{pq}, \tag{1.103}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ *and locally in any space* $B^\sigma_{pq}(\mathbb{R}^n)$ *with* $\sigma < s$. *The representation* (1.103) *is unique,*

$$\lambda_m^{j,G} = 2^{jn/2} \left(f, \Psi^j_{G,m}\right) \tag{1.104}$$

and

$$I : f \mapsto \left\{ 2^{jn/2} \left(f, \Psi^j_{G,m} \right) \right\} \tag{1.105}$$

is an isomorphic map of $B^s_{pq}(\mathbb{R}^n)$ *onto* b^s_{pq}. *If, in addition,* $p < \infty$, $q < \infty$, *then* $\left\{ \Psi^j_{G,m} \right\}$ *is an unconditional basis in* $B^s_{pq}(\mathbb{R}^n)$.

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, *and*

$$u > \max(s, \sigma_{pq} - s). \tag{1.106}$$

Let $f \in S'(\mathbb{R}^n)$. *Then* $f \in F^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^j_{G,m}, \quad \lambda \in f^s_{pq}, \tag{1.107}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ *and locally in any space* $F^\sigma_{pq}(\mathbb{R}^n)$ *with* $\sigma < s$. *The representation* (1.107) *is unique with* (1.104). *Furthermore* I *in* (1.105) *is an isomorphic map of* $F^s_{pq}(\mathbb{R}^n)$ *onto* f^s_{pq}. *If, in addition,* $q < \infty$, *then* $\left\{ \Psi^j_{G,m} \right\}$ *is an unconditional basis in* $F^s_{pq}(\mathbb{R}^n)$.

Proof. Step 1. This theorem coincides with the corresponding assertions in [Tri04] and [T06], Section 3.1.3, Theorem 3.5, pp. 153–156, under the more restrictive smoothness and cancellation assumptions (1.87), (1.88) with

$$u > \max\left(s, \frac{2n}{p} + \frac{n}{2} - s\right) \quad \text{and} \quad u > \max\left(s, \frac{2n}{\min(p,q)} + \frac{n}{2} - s\right) \tag{1.108}$$

for $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, respectively. But all technicalities such as the unconditional convergence of (1.103), (1.107), the uniqueness (1.104), the isomorphic map (1.105), the use of duality, (1.45), (1.46), the unconditional bases, can be taken over verbatim. Hence it remains to justify that the improved assumptions (1.102) and (1.106) are natural and sufficient.

Step 2. Let f be given by (1.103). Then

$$a^G_{jm} = 2^{-j(s-\frac{n}{p})} \, 2^{-jn/2} \, \Psi^j_{G,m}, \quad G \in G^j, \tag{1.109}$$

are atoms in $B^s_{pq}(\mathbb{R}^n)$ according to Definition 1.5 and Theorem 1.7 (i) with $K = L = u$ (up to unimportant constants). Having the different normalisations for b^s_{pq} in (1.97) and for b_{pq} in (1.27) in mind one obtains by Theorem 1.7 (i) that $f \in B^s_{pq}(\mathbb{R}^n)$ and

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\| \le c \, \|\lambda \,|b^s_{pq}\| \tag{1.110}$$

where the extra summation over G does not influence this argument. Here c in (1.110) is independent of f. Conversely, if $f \in B^s_{pq}(\mathbb{R}^n)$, then

$$k^G_{jm} = 2^{jn/2} \, \Psi^j_{G,m}, \quad j \in \mathbb{N}_0, \; G \in G^j, \; m \in \mathbb{Z}^n, \tag{1.111}$$

are kernels of corresponding local means in $B^s_{pq}(\mathbb{R}^n)$ according to Definitions 1.9, 1.13 based on (1.91), and Theorem 1.15 (i) with $A = B = u$ (again neglecting unimportant constants). The modification

$$k(f) = \{k^G_{jm}(f) : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^n\} \tag{1.112}$$

of (1.44) is immaterial for the application of Theorem 1.15 (i). One obtains

$$\|k(f) \,|b^s_{pq}\| \le c \,\|f \,|B^s_{pq}(\mathbb{R}^n)\| \tag{1.113}$$

for some $c > 0$ and all $f \in B^s_{pq}(\mathbb{R}^n)$. By (1.104) one has $k(f) = \lambda$ and one obtains by (1.110) and (1.113) that

$$\|\lambda \,|b^s_{pq}\| \sim \|f \,|B^s_{pq}(\mathbb{R}^n)\|. \tag{1.114}$$

Now one obtains part (i) of the above theorem in the same way as in [Tri04], [T06].

Step 3. The proof for the F-spaces is the same now relying on Theorems 1.7 (ii) and 1.15 (ii). □

Remark 1.21. Again we followed closely [Tri07c].

1.2.3 Wavelets in $A^s_{pq}(\mathbb{R}^n, w)$

Based on [HaT05] we extended in [T06], Section 6.2, Theorem 1.20 with u as in (1.108) to some weighted spaces. There is no difficulty to apply the corresponding arguments to the above improved version with u according to (1.102), (1.106). The main reason for giving an explicit formulation comes from the application to construct wavelet bases for periodic spaces which is the subject of Section 1.3.

Definition 1.22. Let w be a real C^∞ function in $\mathbb{R}^n$ such that for all $\gamma \in \mathbb{N}^n_0$ and suitable positive numbers c_γ,

$$|D^\gamma w(x)| \le c_\gamma \, w(x) \quad \text{for all } x \in \mathbb{R}^n, \tag{1.115}$$

and

$$0 < w(x) \le c\, w(y) \, \big(1 + |x-y|^2\big)^{\alpha/2} \quad \text{for all } x \in \mathbb{R}^n,\ y \in \mathbb{R}^n, \tag{1.116}$$

and some constants $c > 0$ and $\alpha \ge 0$. Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be a resolution of unity according to (1.5)–(1.7).

(i) Let p, q, s as in (1.8). Then $B^s_{pq}(\mathbb{R}^n, w)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|B^s_{pq}(\mathbb{R}^n, w)\| = \Big(\sum_{j=0}^\infty 2^{jsq} \big\| w(\varphi_j \hat{f})^\vee | L_p(\mathbb{R}^n) \big\|^q \Big)^{1/q} < \infty \tag{1.117}$$

(with the usual modification if $q = \infty$).

(ii) Let p, q, s as in (1.10). Then $F^s_{pq}(\mathbb{R}^n, w)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|F^s_{pq}(\mathbb{R}^n, w)\| = \Big\| w \Big(\sum_{j=0}^{\infty} 2^{jsq} \, |(\varphi_j \hat{f})^\vee(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty \tag{1.118}$$

(with the usual modification if $q = \infty$).

Remark 1.23. These weighted spaces (avoiding local singularities) play a role in the theory of function spaces and had been studied in detail in [ET96], Chapter 4 (mainly based on [HaT94a], [HaT94b]) and [T06], Chapter 6, where one finds also references to the literature. Here we restrict our attention to an extension of Theorem 1.20 to these spaces. For this purpose one has first to modify the sequence spaces in Definition 1.18 using the same notation as there.

Definition 1.24. Let w be the same weight as in Definition 1.22 and let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then $b^s_{pq}(w)$ is the collection of all sequences (1.96) such that

$$\|\lambda \,|b^s_{pq}(w)\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G \in G^j} \Big(\sum_{m \in \mathbb{Z}^n} w(2^{-j}m)^p \, |\lambda^{j,G}_m|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{1.119}$$

and $f^s_{pq}(w)$ is the collection of all sequences (1.96) such that

$$\|\lambda \,| f^s_{pq}(w)\| = \Big\| \Big(\sum_{j,G,m} 2^{jsq} \, w(2^{-j}m)^q \, |\lambda^{j,G}_m \chi_{jm}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty \tag{1.120}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

Remark 1.25. This extends Definition 1.18 to the weighted case and coincides with [T06], Definition 6.11. We extend now Theorem 1.20 to the weighted case in the same understanding as described there now with respect to the spaces $A^s_{pq}(\mathbb{R}^n, w)$ where either $A = B$ or $A = F$.

Theorem 1.26. *Let $A^s_{pq}(\mathbb{R}^n, w)$ with $A = B$ or $A = F$ be the spaces as introduced in Definition 1.22 and let $a^s_{pq}(w)$ be the related sequence spaces according to Definition 1.24.*

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and let $\Psi^j_{G,m}$ be the wavelets (1.91) based on (1.87), (1.88) with*

$$u > \max(s, \sigma_p - s). \tag{1.121}$$

Let $f \in S'(\mathbb{R}^n)$. Then $f \in B^s_{pq}(\mathbb{R}^n, w)$ if, and only if, it can be represented as

$$f = \sum_{j,G,m} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^j_{G,m}, \quad \lambda \in b^s_{pq}(w), \tag{1.122}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ *and in any space* $B^\sigma_{pq}(\mathbb{R}^n, \widetilde{w})$ *with* $\sigma < s$ *and* $\widetilde{w}(x)w^{-1}(x) \to 0$ *if* $|x| \to \infty$. *Furthermore, the representation* (1.122) *is unique,*

$$\lambda^{j,G}_m = 2^{jn/2} \left(f, \Psi^j_{G,m}\right) \tag{1.123}$$

and

$$I : f \mapsto \left\{2^{jn/2} \left(f, \Psi^j_{G,m}\right)\right\} \tag{1.124}$$

is an isomorphic map of $B^s_{pq}(\mathbb{R}^n, w)$ *onto* $b^s_{pq}(w)$. *If, in addition,* $p < \infty$, $q < \infty$, *then* $\left\{\Psi^j_{G,m}\right\}$ *is an unconditional basis in* $B^s_{pq}(\mathbb{R}^n, w)$.

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ *and*

$$u > \max(s, \sigma_{pq} - s). \tag{1.125}$$

Let $f \in S'(\mathbb{R}^n)$. *Then* $f \in F^s_{pq}(\mathbb{R}^n, w)$ *if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^j_{G,m}, \quad \lambda \in f^s_{pq}(w), \tag{1.126}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ *and in any space* $F^\sigma_{pq}(\mathbb{R}^n, \widetilde{w})$ *with* $\sigma < s$ *and* $\widetilde{w}(x)w^{-1}(x) \to 0$ *if* $|x| \to \infty$. *Furthermore, the representation* (1.126) *is unique with* (1.123) *and* I *in* (1.124) *is an isomorphic map of* $F^s_{pq}(\mathbb{R}^n, w)$ *onto* $f^s_{pq}(w)$. *If, in addition,* $q < \infty$, *then* $\left\{\Psi^j_{G,m}\right\}$ *is an unconditional basis in* $F^s_{pq}(\mathbb{R}^n, w)$.

Proof. This theorem with u as in (1.108) coincides with [T06], Theorem 6.15, pp. 270–71. The corresponding proof reduces the weighted case to the unweighted one using localisation and interpolation. But this works also for u as in (1.121) and (1.125) relying now on Theorem 1.20. □

1.3 Periodic spaces on $\mathbb{R}^n$ and $\mathbb{T}^n$

1.3.1 Definitions and basic properties

Let

$$\mathbb{T}^n = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \ 0 \le x_j \le 1\right\} \tag{1.127}$$

be the n-torus (adapted to wavelets and $\mathbb{Z}^n$) where opposite points are identified in the usual way. Then periodic distributions $f \in D'(\mathbb{T}^n)$ can be represented as

$$f = \sum_{m \in \mathbb{Z}^n} a_m \, e^{i2\pi mx}, \quad x \in \mathbb{T}^n, \tag{1.128}$$

where the Fourier coefficients $\{a_m\} \subset \mathbb{C}$ are of at most polynomial growth,

$$|a_m| \le c\,(1 + |m|)^\varkappa \quad \text{for some } c > 0,\ \varkappa > 0 \text{ and all } m \in \mathbb{Z}^n. \tag{1.129}$$

The theory of periodic distributions and related periodic spaces $B^s_{pq}(\mathbb{T}^n)$ and $F^s_{pq}(\mathbb{T}^n)$ has some history which is not the subject of what follows. We rely on [ST87], Chapter 3, and [T83], Chapter 9, with a reference to [Tri83]. We wish to complement these considerations by the periodic counterpart of the wavelet isomorphism according to Theorem 1.20. First we collect what we need in the sequel.

We extend $f \in D'(\mathbb{T}^n)$ given by (1.128) with (1.129) from $D'(\mathbb{T}^n)$ to $S'(\mathbb{R}^n)$,

$$\mathrm{ext}^{\mathrm{per}} f = \sum_{m\in\mathbb{Z}^n} a_m \, e^{i2\pi mx}, \quad x \in \mathbb{R}^n. \tag{1.130}$$

Then $\mathrm{ext}^{\mathrm{per}} f \in S'^{\mathrm{per}}(\mathbb{R}^n)$, where

$$S'^{\mathrm{per}}(\mathbb{R}^n) = \left\{h \in S'(\mathbb{R}^n) : \; h(\cdot - k) = h \text{ for all } k \in \mathbb{Z}^n\right\} \tag{1.131}$$

are the periodic distributions on $\mathbb{R}^n$. Recall that $h \in S'^{\mathrm{per}}(\mathbb{R}^n)$ if, and only if, h can be represented as

$$h = \sum_{m\in\mathbb{Z}^n} a_m \, e^{i2\pi mx}, \quad x \in \mathbb{R}^n, \tag{1.132}$$

with (1.129) for some $c > 0$ and $\varkappa > 0$. We have

$$\begin{aligned} (Fh)(\xi) &= \sum_{m\in\mathbb{Z}^n} a_m \, F\left(e^{i2\pi xm}\right)(\xi) \\ &= \sum_{m\in\mathbb{Z}^n} a_m \, (F1)(\xi - 2\pi m) \\ &= \sum_{m\in\mathbb{Z}^n} (2\pi)^{n/2} \, a_m \, \delta_{2\pi m}, \end{aligned} \tag{1.133}$$

where $\delta_{2\pi m}$ is the δ-distribution with respect to the off-point $2\pi m$. Let $\{\varphi_j\}$ be the same dyadic resolution of unity as in Definition 1.1. Then

$$\varphi_j \hat{h} = \sum_{m\in\mathbb{Z}^n} a_m \, (2\pi)^{n/2} \, \varphi_j(2\pi m) \, \delta_{2\pi m} \tag{1.134}$$

and

$$\left(\varphi_j \hat{h}\right)^{\vee}(x) = \sum_{m\in\mathbb{Z}^n} a_m \, \varphi_j(2\pi m) \, e^{i2\pi mx}. \tag{1.135}$$

Let

$$w_\gamma(x) = (1 + |x|^2)^{\gamma/2}, \quad \gamma \in \mathbb{R}, \tag{1.136}$$

be an admitted weight according to Definition 1.22 (with $\alpha = |\gamma|$) and let $A^s_{pq}(\mathbb{R}^n, w_\gamma)$ be the corresponding spaces. Let

$$A^{s,\mathrm{per}}_{pq}(\mathbb{R}^n, w_\gamma) = A^s_{pq}(\mathbb{R}^n, w_\gamma) \cap S'^{\mathrm{per}}(\mathbb{R}^n). \tag{1.137}$$

If $h \in S'^{\mathrm{per}}(\mathbb{R}^n)$ is given by (1.132) then one has by (1.135) and (1.117), (1.118) that

$$\|h \,|B^{s,\mathrm{per}}_{pq}(\mathbb{R}^n, w_\gamma)\| = \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big\| w_\gamma(x) \sum_{m\in\mathbb{Z}^n} a_m \, \varphi_j(2\pi m)\, e^{i2\pi mx} |L_p(\mathbb{R}^n)\Big\|^q \Big)^{1/q} \tag{1.138}$$

and

$$\|h \,|F^{s,\mathrm{per}}_{pq}(\mathbb{R}^n, w_\gamma)\| = \Big\| w_\gamma(x) \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big| \sum_{m\in\mathbb{Z}^n} a_m \, \varphi_j(2\pi m)\, e^{i2\pi mx} \Big|^q \Big)^{1/q} |L_p(\mathbb{R}^n)\Big\|. \tag{1.139}$$

These spaces are non-trivial if, and only if, $w_\gamma \in L_p(\mathbb{R}^n)$.

The spaces $A^s_{pq}(\mathbb{T}^n)$ will be introduced in the standard way where we refer for further details to the above literature, especially to [ST87], Chapter 3.

Definition 1.27. Let $\varphi = \{\varphi_j\}$ be again the above dyadic resolution of unity. Let $s \in \mathbb{R}$ and $0 < q \le \infty$.

(i) Let $0 < p \le \infty$. Then

$$B^s_{pq}(\mathbb{T}^n) = \big\{ f \in D'(\mathbb{T}^n) : \ \|f \,|B^s_{pq}(\mathbb{T}^n)\|_\varphi < \infty \big\}, \tag{1.140}$$

$$\|f \,|B^s_{pq}(\mathbb{T}^n)\|_\varphi = \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big\| \sum_{m\in\mathbb{Z}^n} a_m \, \varphi_j(2\pi m)\, e^{i2\pi mx} \,|L_p(\mathbb{T}^n)\Big\|^q \Big)^{1/q} \tag{1.141}$$

where f is given by (1.128).

(ii) Let $0 < p < \infty$. Then

$$F^s_{pq}(\mathbb{T}^n) = \big\{ f \in D'(\mathbb{T}^n) : \ \|f \,|F^s_{pq}(\mathbb{T}^n)\|_\varphi < \infty \big\}, \tag{1.142}$$

$$\|f \,|F^s_{pq}(\mathbb{T}^n)\|_\varphi = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big| \sum_{m\in\mathbb{Z}^n} a_m \, \varphi_j(2\pi m)\, e^{i2\pi mx} \Big|^q \Big)^{1/q} |L_p(\mathbb{T}^n)\Big\| \tag{1.143}$$

where f is given by (1.128).

Remark 1.28. This coincides with [ST87], Definition, pp. 162–63. Basic assertions parallel to the corresponding theory on $\mathbb{R}^n$ may be found in [ST87], Section 3.5. In particular, the above spaces are independent of admitted φ. Furthermore,

$$W^k_p(\mathbb{T}^n) = F^k_{p,2}(\mathbb{T}^n), \quad k \in \mathbb{N}_0, \ 1 < p < \infty, \tag{1.144}$$

are the usual periodic Sobolev spaces and

$$\mathcal{C}^s(\mathbb{T}^n) = B^s_{\infty\infty}(\mathbb{T}^n), \quad s > 0, \tag{1.145}$$

are the usual periodic Hölder–Zygmund spaces. All classical spaces on $\mathbb{R}^n$ of Sobolev–Besov type as discussed in Remark 1.2 have periodic counterparts and related intrinsic norms. A corresponding list may be found in [ST87], Section 3.5.4, pp. 167–69. By the support properties of φ_j the sums over $\mathbb{Z}^n$ in (1.141), (1.143) and in (1.138), (1.139) are trigonometrical polynomials and the corresponding quasi-norms make sense. They are connected by the extension operator $\mathrm{ext}^{\mathrm{per}}$ from $D'(\mathbb{T}^n)$ into $S'^{\mathrm{per}}(\mathbb{R}^n)$ according to (1.130).

Theorem 1.29. *Let $s \in \mathbb{R}$, $0 < p \le \infty$ ($p < \infty$ for the F-spaces), $0 < q \le \infty$. Let w_γ be as in (1.136) with $\gamma < -\frac{n}{p}$. Let $A^{s,\mathrm{per}}_{pq}(\mathbb{R}^n, w_\gamma)$ be as in (1.137). Then $\mathrm{ext}^{\mathrm{per}}$ is an isomorphic map from*

$$B^s_{pq}(\mathbb{T}^n) \quad \text{onto} \quad B^{s,\mathrm{per}}_{pq}(\mathbb{R}^n, w_\gamma) \tag{1.146}$$

and from

$$F^s_{pq}(\mathbb{T}^n) \quad \text{onto} \quad F^{s,\mathrm{per}}_{pq}(\mathbb{R}^n, w_\gamma). \tag{1.147}$$

The spaces on the right-hand sides of (1.146), (1.147) are independent of γ.

Proof. Obviously,

$$\mathbb{R}^n = \bigcup_{l \in \mathbb{Z}^n} Q_l, \quad Q_l = \left\{ x \in \mathbb{R}^n : l_j \le x_j \le l_{j+1} \right\}. \tag{1.148}$$

Then one obtains for $p < \infty$,

$$\begin{aligned} &\Big\| w_\gamma(x) \sum_{m \in \mathbb{Z}^n} a_m \, \varphi_j(2\pi m) \, e^{i2\pi m x} \, | L_p(\mathbb{R}^n) \Big\|^p \\ &\quad = \Big\| \sum_{m \in \mathbb{Z}^n} a_m \, \varphi_j(2\pi m) \, e^{i2\pi m x} \, | L_p(\mathbb{T}^n) \Big\|^p \int_{\mathbb{R}^n} (1 + |x|^2)^{\gamma p/2} \, \mathrm{d}x \\ &\quad \sim \Big\| \sum_{m \in \mathbb{Z}^n} a_m \, \varphi_j(2\pi m) \, e^{i2\pi m x} \, | L_p(\mathbb{T}^n) \Big\|^p. \end{aligned} \tag{1.149}$$

This proves (1.146) (modification if $p = \infty$). Similarly one obtains (1.147). □

Remark 1.30. The above isomorphism had been used in [T83], Section 9, and [Tri83] to transfer properties of function spaces on $\mathbb{R}^n$ to the corresponding spaces on $\mathbb{T}^n$. The spaces on the right-hand sides of (1.146), (1.147) are independent of γ. This suggests the following definition.

Definition 1.31. Let $S'^{\mathrm{per}}(\mathbb{R}^n)$ be as in (1.131). Let $s \in \mathbb{R}$, $0 < p \le \infty$ ($p < \infty$ for the F-spaces), $0 < q \le \infty$ and $\gamma < -\frac{n}{p}$. Then

$$B^{s,\mathrm{per}}_{pq}(\mathbb{R}^n) = B^{s,\mathrm{per}}_{pq}(\mathbb{R}^n, w_\gamma) = B^s_{pq}(\mathbb{R}^n, w_\gamma) \cap S'^{\mathrm{per}}(\mathbb{R}^n) \tag{1.150}$$

and

$$F^{s,\mathrm{per}}_{pq}(\mathbb{R}^n) = F^{s,\mathrm{per}}_{pq}(\mathbb{R}^n, w_\gamma) = F^s_{pq}(\mathbb{R}^n, w_\gamma) \cap S'^{\mathrm{per}}(\mathbb{R}^n). \tag{1.151}$$

1.3.2 Wavelets in $A^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$

Let $L \in \mathbb{N}$. As a consequence of the multi-resolution analysis one can replace ψ_F and ψ_M in (1.87), (1.88) by

$$\psi_F^L(\cdot) = \psi_F(2^L \cdot) \in C^u(\mathbb{R}), \quad \psi_M^L(\cdot) = \psi_M(2^L \cdot) \in C^u(\mathbb{R}), \quad u \in \mathbb{N}, \tag{1.152}$$

and $\Psi^j_{G,m}$ in (1.91) by

$$\Psi^{j,L}_{G,m}(x) = 2^{(j+L)n/2} \prod_{r=1}^n \psi_{G_r}\big(2^{j+L}x_r - m_r\big), \quad G \in G^j, \ m \in \mathbb{Z}^n. \tag{1.153}$$

We choose (and fix) L such that

$$\operatorname{supp} \Psi^{0,L}_{G,0} \subset \{x \in \mathbb{R}^n : \ |x| < 1/2\}, \quad G \in G^0 = \{F, M\}^n. \tag{1.154}$$

Since $L \in \mathbb{N}$ is fixed once and for all in Section 1.3 we simplify our notation and write $\Psi^j_{G,m}$ instead of $\Psi^{j,L}_{G,m}$ given by (1.152), (1.153). Then one has

$$\operatorname{supp} \Psi^j_{G,0} = \operatorname{supp} \Psi^0_{G,0}(2^j \cdot) \subset \big\{x \in \mathbb{R}^n : \ |x| < 2^{-j-1}\big\} \tag{1.155}$$

for $j \in \mathbb{N}_0$ and $G \in G^j$. The adapted wavelet expansions (1.93), (1.94) read now as follows,

$$f = \sum_{j=0}^\infty \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G}_m \, 2^{-(j+L)n/2} \, \Psi^j_{G,m} \tag{1.156}$$

with

$$\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{(j+L)n/2} \big(f, \Psi^j_{G,m}\big), \tag{1.157}$$

based on the newly interpreted orthonormal basis in (1.92). Let

$$\mathbb{P}^n_j = \big\{m \in \mathbb{Z}^n : \ 0 \le m_r < 2^{j+L}\big\}, \quad j \in \mathbb{N}_0, \tag{1.158}$$

be the $2^{(j+L)n}$ lattice points in $2^{(j+L)}\mathbb{T}^n$. Then

$$\Psi^j_{G,m,\mathrm{per}}(x) = \sum_{l \in \mathbb{Z}^n} \Psi^j_{G,m}(x - l) = \sum_{l \in \mathbb{Z}^n} \Psi^j_{G,m+2^{j+L}l}(x) \tag{1.159}$$

with $j \in \mathbb{N}_0$ and $m \in \mathbb{P}^n_j$ are the periodic extensions of the distinguished wavelets with off-points $2^{-j-L}m \in \mathbb{T}^n$. With $u \in \mathbb{N}$ as in (1.87), (1.88) one has

$$\Psi^j_{G,m,\mathrm{per}} \in C^u(\mathbb{R}^n) \cap A^{t,\mathrm{per}}_{pq}(\mathbb{R}^n), \quad t < u, \tag{1.160}$$

with $A = B$ or $A = F$ according to Definition 1.31. In particular it makes sense to ask whether these periodic wavelets may serve as a basis in the spaces $A^{t,\mathrm{per}}_{pq}(\mathbb{R}^n)$ in analogy to Theorem 1.26. First we need the corresponding sequence spaces.

Definition 1.32. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then $b^{s,\text{per}}_{pq}$ is the collection of all sequences

$$\mu = \{\mu^{j,G}_m \in \mathbb{C} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{P}^n_j\} \tag{1.161}$$

such that

$$\|\mu \,|b^{s,\text{per}}_{pq}\| = \Big(\sum_{j=0}^\infty 2^{j(s-\frac{n}{p})q} \sum_{G \in G^j} \Big(\sum_{m \in \mathbb{P}^n_j} |\mu^{j,G}_m|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{1.162}$$

and f^s_{pq} is the collection of all sequences (1.161) such that

$$\|\mu | f^{s,\text{per}}_{pq}\| = \Big\| \Big(\sum_{j,G,m} 2^{jsq} \,|\mu^{j,G}_m \, \chi_{jm}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{T}^n) \Big\| < \infty \tag{1.163}$$

with the usual modification if $p = \infty$ and/or $q = \infty$, where χ_{jm} is the characteristic function of a cube with the left corner $2^{-j-L}m$ and of side-length 2^{-j-L} (a subcube of $\mathbb{T}^n$).

Remark 1.33. This is the periodic counterpart of Definitions 1.18, 1.24. With $a = b$ or $a = f$ let $a^s_{pq}(w)$ and $a^{s,\text{per}}_{pq}$ be the corresponding sequence spaces. Let

$$\mu^{j,G}_m = \lambda^{j,G}_m = \lambda^{j,G}_{m+2^{j+L}l} \quad \text{for } j \in \mathbb{N}_0,\ m \in \mathbb{P}^n_j,\ l \in \mathbb{Z}^n. \tag{1.164}$$

Then one obtains

$$\|\mu \,|a^{s,\text{per}}_{pq}\| \sim \|\lambda \,|a^s_{pq}(w_\gamma)\|, \quad \gamma < -\tfrac{n}{p}, \tag{1.165}$$

in the same way as in (1.149). For the corresponding restriction $\Psi^{j,\text{per}}_{G,m}$,

$$\Psi^{j,\text{per}}_{G,m}(x) = \Psi^j_{G,m,\text{per}}(x), \quad x \in \mathbb{T}^n, \tag{1.166}$$

of the periodic wavelets in (1.159), (1.160), one has the following basic assertion.

Proposition 1.34. *Let $u \in \mathbb{N}$. Then*

$$\{\Psi^{j,\text{per}}_{G,m} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{P}^n_j\} \tag{1.167}$$

is an orthonormal basis in $L_2(\mathbb{T}^n)$.

Proof. From the corresponding assertion for $\Psi^j_{G,m}$ in (1.92), now modified by (1.153), and (1.155) it follows that the functions in (1.167) are orthonormal. Let $f \in L_2(\mathbb{T}^n)$. Then the restriction to $\mathbb{T}^n$ of the expansion of

$$g = \text{ext}^{\text{per}} f \in L_2(\mathbb{R}^n, w_\gamma), \quad \gamma < -\tfrac{n}{2}, \tag{1.168}$$

according to Theorem 1.26 is an expansion of f by the system (1.167). (One may also consult (1.176) below.) Hence the system (1.167) is complete. □

Remark 1.35. This is the periodic counterpart of (1.92)–(1.94). The periodic counterpart of Theorem 1.20 is the subject of Section 1.3.3. First we deal with the periodic spaces in $\mathbb{R}^n$ according to Definition 1.31. Let

$$\big(f, \Psi^{j,\mathrm{per}}_{G,m}\big)_\pi = \int_{\mathbb{T}^n} f(x)\, \Psi^{j,\mathrm{per}}_{G,m}(x)\, \mathrm{d}x \tag{1.169}$$

be the usual dual pairing in $(D(\mathbb{T}^n), D'(\mathbb{T}^n))$, appropriately interpreted. (Recall that the functions $\Psi^{j,\mathrm{per}}_{G,m}$ are real.)

Theorem 1.36. *Let $\Psi^j_{G,m,\mathrm{per}}$ with $j \in \mathbb{N}_0$, $G \in G^j$ and $m \in \mathbb{P}^n_j$ be the periodic wavelets (1.159) in $\mathbb{R}^n$ with $u \in \mathbb{N}$ in (1.160).*

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and*

$$u > \max(s, \sigma_p - s). \tag{1.170}$$

Let $f \in S'(\mathbb{R}^n)$. Then $f \in B^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j,G,m} \mu^{j,G}_m\, 2^{-(j+L)n/2}\, \Psi^j_{G,m,\mathrm{per}}, \quad \mu \in b^{s,\mathrm{per}}_{pq}, \tag{1.171}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ and in any space $B^\sigma_{pq}(\mathbb{R}^n, w_\gamma)$ with $\sigma < s$ and $\gamma < -\frac{n}{p}$. Furthermore, this representation is unique,

$$\mu^{j,G}_m = 2^{(j+L)n/2}\, \big(f, \Psi^{j,\mathrm{per}}_{G,m}\big)_\pi \tag{1.172}$$

according to (1.166), (1.169), and

$$I: f \mapsto \big\{2^{(j+L)n/2}\big(f, \Psi^{j,\mathrm{per}}_{G,m}\big)_\pi\big\} \tag{1.173}$$

is an isomorphic map of $B^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$ onto $b^{s,\mathrm{per}}_{pq}$. If, in addition, $p < \infty$, $q < \infty$, then $\big\{\Psi^j_{G,m,\mathrm{per}}\big\}$ is an unconditional basis in $B^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$.

(ii) *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and*

$$u > \max(s, \sigma_{pq} - s). \tag{1.174}$$

Let $f \in S'(\mathbb{R}^n)$. Then $f \in F^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j,G,m} \mu^{j,G}_m\, 2^{-(j+L)n/2}\, \Psi^j_{G,m,\mathrm{per}}, \quad \mu \in f^{s,\mathrm{per}}_{pq}, \tag{1.175}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ and in any space $F^\sigma_{pq}(\mathbb{R}^n, w_\gamma)$ with $\sigma < s$ and $\gamma < -\frac{n}{p}$. Furthermore, this representation is unique with (1.172) and I in (1.173) is an isomorphic map of $F^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$ onto $f^{s,\mathrm{per}}_{pq}$. If, in addition, $q < \infty$, then $\big\{\Psi^j_{G,m,\mathrm{per}}\big\}$ is an unconditional basis in $F^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$.

Proof. Step 1. If $f \in S'(\mathbb{R}^n)$ is represented by (1.171) or (1.175) then it follows from (1.164), (1.165) and Theorem 1.26 that $f \in A^s_{pq}(\mathbb{R}^n, w_\gamma)$ with $\gamma < -\frac{n}{p}$. Furthermore, $f \in S'^{\mathrm{per}}(\mathbb{R}^n)$ and, hence, $f \in A^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$ according to Definition 1.31.

Step 2. If $f \in A^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$ then one has for $j \in \mathbb{N}_0$, $G \in G^j$, $m \in \mathbb{P}^n_j$ and $l \in \mathbb{Z}^n$ that (in the usual interpretation)

$$
\begin{aligned}
\big(f, \Psi^j_{G,m+2^{j+L}l}\big) &= \int_{\mathbb{R}^n} f(x)\,\Psi^j_{G,m}(x-l)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} f(x-l)\,\Psi^j_{G,m}(x-l)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} f(x)\,\Psi^j_{G,m}(x)\,\mathrm{d}x \\
&= \int_{\mathbb{T}^n} f(x)\,\Psi^{j,\mathrm{per}}_{G,m}(x)\,\mathrm{d}x \\
&= \big(f, \Psi^{j,\mathrm{per}}_{G,m}\big)_\pi
\end{aligned}
\tag{1.176}
$$

where we used that f is periodic in $\mathbb{R}^n$ and (1.159), (1.166), (1.169). One can justify the formal calculation in (1.176) by duality as in $\mathbb{R}^n$ and a limiting argument approximating f by trigonometrical polynomials. Expanding now f according to Theorem 1.26 than one obtains by (1.176), (1.165) the expansions (1.171), (1.175) with (1.172). □

1.3.3 Wavelets in $A^s_{pq}(\mathbb{T}^n)$

By Proposition 1.34 the system $\{\Psi^{j,\mathrm{per}}_{G,m}\}$ in (1.167) is an orthonormal basis in $L_2(\mathbb{T}^n)$. By Theorem 1.29,

$$
\mathrm{ext}^{\mathrm{per}}\, A^s_{pq}(\mathbb{T}^n) = A^{s,\mathrm{per}}_{pq}(\mathbb{R}^n) \tag{1.177}
$$

is an isomorphic map for all admitted parameters. This gives the possibility to transfer Theorem 1.36 from the periodic spaces on $\mathbb{R}^n$ to the corresponding spaces on $\mathbb{T}^n$.

Theorem 1.37. *Let $\{\Psi^{j,\mathrm{per}}_{G,m}\}$ be the orthonormal basis in $L_2(\mathbb{T}^n)$ according to Proposition 1.34 with $u \in \mathbb{N}$ as in (1.160).*

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and*

$$
u > \max(s, \sigma_p - s). \tag{1.178}
$$

Let $f \in D'(\mathbb{T}^n)$. Then $f \in B^s_{pq}(\mathbb{T}^n)$ if, and only if, it can be represented as

$$
f = \sum_{j,G,m} \mu^{j,G}_m\, 2^{-(j+L)n/2}\, \Psi^{j,\mathrm{per}}_{G,m}, \quad \mu \in b^{s,\mathrm{per}}_{pq}, \tag{1.179}
$$

unconditional convergence being in $D'(\mathbb{T}^n)$ and in any space $B^\sigma_{pq}(\mathbb{T}^n)$ with $\sigma < s$. Furthermore, this representation is unique,

$$
\mu^{j,G}_m = 2^{(j+L)n/2}\,\big(f, \Psi^{j,\mathrm{per}}_{G,m}\big)_\pi, \tag{1.180}
$$

and

$$I: f \mapsto \left\{2^{(j+L)n/2} \left(f, \Psi^{j,\mathrm{per}}_{G,m}\right)_\pi\right\} \tag{1.181}$$

is an isomorphic map of $B^s_{pq}(\mathbb{T}^n)$ *onto* $b^{s,\mathrm{per}}_{pq}$. *If, in addition,* $p < \infty$, $q < \infty$, *then* $\{\Psi^{j,\mathrm{per}}_{G,m}\}$ *is an unconditional basis in* $B^s_{pq}(\mathbb{T}^n)$.

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, *and*

$$u > \max(s, \sigma_{pq} - s). \tag{1.182}$$

Let $f \in D'(\mathbb{T}^n)$. *Then* $f \in F^s_{pq}(\mathbb{T}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \mu^{j,G}_m \, 2^{-(j+L)n/2} \, \Psi^{j,\mathrm{per}}_{G,m}, \quad \mu \in f^{s,\mathrm{per}}_{pq}, \tag{1.183}$$

unconditional convergence being in $D'(\mathbb{T}^n)$ *and in any space* $F^\sigma_{pq}(\mathbb{T}^n)$ *with* $\sigma < s$. *Furthermore, this representation is unique with* (1.180) *and* I *in* (1.181) *is an isomorphic map of* $F^s_{pq}(\mathbb{T}^n)$ *onto* $f^{s,\mathrm{per}}_{pq}$. *If, in addition,* $q < \infty$, *then* $\{\Psi^{j,\mathrm{per}}_{G,m}\}$ *is an unconditional basis in* $F^s_{pq}(\mathbb{T}^n)$.

Remark 1.38. As mentioned this assertion is an immediate consequence of Theorem 1.36. We used the same notation as there. The idea to periodise wavelet bases on $\mathbb{R}$ to construct (periodic) wavelet bases on the 1-torus $\mathbb{T}$, preferably for $L_2(\mathbb{T})$ appears in the literature several times. We refer to [Dau92], pp. 304–05, [Mal99], Section 7.5.1, pp. 282–83, and [Woj97], Section 2.5, pp. 38–42. Although there is a huge literature about periodic wavelets we could not find papers dealing with wavelet bases in the above setting. We followed here essentially [Tri07c].

Chapter 2
Spaces on arbitrary domains

2.1 Basic definitions

2.1.1 Function spaces

We fix some notation. Let Ω be an arbitrary domain in $\mathbb{R}^n$. Domain means open set. Then $L_p(\Omega)$ with $0 < p \le \infty$ is the standard quasi-Banach space of all complex-valued Lebesgue measurable functions f in Ω such that

$$\|f \,|L_p(\Omega)\| = \Big(\int_\Omega |f(x)|^p \,\mathrm{d}x \Big)^{1/p} \tag{2.1}$$

(with the natural modification if $p = \infty$) is finite. As usual, $D(\Omega) = C_0^\infty(\Omega)$ stands for the collection of all complex-valued infinitely differentiable functions in $\mathbb{R}^n$ with compact support in Ω. Let $D'(\Omega)$ be the dual space of all distributions in Ω. Let $g \in S'(\mathbb{R}^n)$. Then we denote by $g|\Omega$ its *restriction* to Ω,

$$g|\Omega \in D'(\Omega): \quad (g|\Omega)(\varphi) = g(\varphi) \quad \text{for } \varphi \in D(\Omega). \tag{2.2}$$

With $A = B$ or $A = F$ the spaces $A^s_{pq}(\mathbb{R}^n)$ have the same meaning as in Definition 1.1

Definition 2.1. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \ne \mathbb{R}^n$ and let

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}, \tag{2.3}$$

with $p < \infty$ for the F-spaces.

(i) Then

$$A^s_{pq}(\Omega) = \big\{ f \in D'(\Omega): \ f = g|\Omega \text{ for some } g \in A^s_{pq}(\mathbb{R}^n)\big\}, \tag{2.4}$$

$$\|f \,|A^s_{pq}(\Omega)\| = \inf \|g \,|A^s_{pq}(\mathbb{R}^n)\|, \tag{2.5}$$

where the infimum is taken over all $g \in A^s_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$.

(ii) Let

$$\tilde{A}^s_{pq}(\bar{\Omega}) = \big\{ f \in A^s_{pq}(\mathbb{R}^n): \ \operatorname{supp} f \subset \bar{\Omega}\big\}. \tag{2.6}$$

Then

$$\tilde{A}^s_{pq}(\Omega) = \big\{ f \in D'(\Omega): \ f = g|\Omega \text{ for some } g \in \tilde{A}^s_{pq}(\bar{\Omega})\big\}, \tag{2.7}$$

$$\|f \,|\tilde{A}^s_{pq}(\Omega)\| = \inf \|g \,|A^s_{pq}(\mathbb{R}^n)\| \tag{2.8}$$

where the infimum is taken over all $g \in \tilde{A}^s_{pq}(\bar{\Omega})$ with $g|\Omega = f$.

Remark 2.2. Part (i) is the usual definition of $A^s_{pq}(\Omega)$ by restriction. The spaces $\tilde{A}^s_{pq}(\bar{\Omega})$ are closed subspaces of $A^s_{pq}(\mathbb{R}^n)$. One has an one-to-one correspondence between $\tilde{A}^s_{pq}(\bar{\Omega})$ and $\tilde{A}^s_{pq}(\Omega)$, written in a somewhat sloppy way as

$$\tilde{A}^s_{pq}(\bar{\Omega}) = \tilde{A}^s_{pq}(\Omega), \tag{2.9}$$

if, and only if,

$$\{h \in A^s_{pq}(\mathbb{R}^n) : \operatorname{supp} h \subset \partial\Omega\} = \{0\}. \tag{2.10}$$

This is the case if Ω is a bounded Lipschitz domain, $0 < p, q \le \infty$ and $s > \sigma_p$, (1.32), since $|\partial\Omega| = 0$ and $A^s_{pq}(\mathbb{R}^n) \subset L^{\mathrm{loc}}_1 \mathbb{R}^n)$. The spaces $\tilde{A}^s_{pq}(\Omega)$ with $s < 0$ will be not of interest for us. We remark that (2.10) does not hold if $s < n\big(\frac{1}{p} - 1\big)$ because δ and its translates are elements of $A^s_{pq}(\mathbb{R}^n)$. But for general domains Ω it may happen that (2.10) is not valid even for spaces $A^s_{pq}(\mathbb{R}^n)$ with $1 \le p, q \le \infty$ and $s > 0$. We give an example looking for spaces on domains Ω with $|\Gamma| > 0$, where $\Gamma = \partial\Omega$. Let $\{r_l : l \in \mathbb{N}\}$ be the set of all rational numbers with $0 < r_l < 1$ and let I_l be open intervals centred at r_l such that $I_l \subset (0, 1)$. Let

$$\Omega = \bigcup_{l=1}^{\infty} I_l \quad \text{with} \ \sum_{l=1}^{\infty} |I_l| < 1. \tag{2.11}$$

Then

$$\Gamma = [0, 1] \backslash \bigcup_{l=1}^{\infty} I_l \quad \text{with } |\Gamma| > 0. \tag{2.12}$$

Let

$$1 < p < \infty, \quad 1 \le q \le \infty, \quad 0 < s < 1/p. \tag{2.13}$$

Then $\chi_l \in F^s_{pq}(\mathbb{R})$ where χ_l is the characteristic function of I_l and one obtains by the homogeneity properties of $F^s_{pq}(\mathbb{R})$ according to [T01], Corollary 5.16, p. 66 (also discussed in Section 2.2 below, especially (2.46)) that

$$\|\chi_l \,|F^s_{pq}(\mathbb{R})\| \sim |I_l|^{\frac{1}{p} - s}, \quad l \in \mathbb{N}. \tag{2.14}$$

Let χ_Ω be the characteristic function of Ω. Then one has by (2.14) that

$$\|\chi_\Omega \,|F^s_{pq}(\mathbb{R})\| \le c \sum_{l=1}^{\infty} |I_l|^{\frac{1}{p} - s} < \infty \tag{2.15}$$

if the $|I_l|$ are chosen sufficiently small. Hence $\chi_\Omega \in F^s_{pq}(\mathbb{R})$ and also $\chi_\Gamma \in F^s_{pq}(\mathbb{R})$. This disproves (2.10). We refer in this context also to [T06], Section 1.11.6, where we discussed assertions of type (2.9) in the case of bounded Lipschitz domains.

Remark 2.3. Definition 2.1 covers as special cases of $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ the corresponding restrictions of the classical spaces discussed in Remark 1.2. This is obvious for the Lebesgue spaces

$$L_p(\Omega) = F^0_{p,2}(\Omega), \quad 1 < p < \infty. \tag{2.16}$$

But for the other classical Sobolev–Besov spaces some care is necessary, in particular in connection with intrinsic norms. This is closely related to the extension problem which is the subject of Chapter 4. In case of the classical Sobolev spaces $W^k_p(\Omega)$ we refer to Definition 2.53 and Remark 2.54 below where we discuss these questions in some detail. Otherwise we remind the reader of some intrinsic norms for the classical Besov spaces

$$B^s_{pq}(\Omega), \quad 1 \le p \le \infty, \ 1 \le q \le \infty, \ s > 0, \tag{2.17}$$

in bounded Lipschitz domains Ω (formally introduced in Definition 3.4 below). Let $\Delta^l_h f$ be the differences in $\mathbb{R}^n$ according to (1.21) where $l \in \mathbb{N}$ and $h \in \mathbb{R}^n$. Let for $x \in \Omega$,

$$(\Delta^l_{h,\Omega} f)(x) = \begin{cases} (\Delta^l_h f)(x) & \text{if } x + kh \in \Omega \text{ for } k = 0, \dots, l, \\ 0 & \text{otherwise,} \end{cases} \tag{2.18}$$

be the differences adapted to Ω. Then $B^s_{pq}(\Omega)$ with p, q, s as in (2.17) can be equivalently normed by

$$\|f \,|B^s_{pq}(\Omega)\|^*_m = \|f \,|L_p(\Omega)\| + \left(\int_0^1 t^{-sq} \sup_{|h| \le t} \|\Delta^m_{h,\Omega} f \,|L_p(\Omega)\|^q \, \frac{\mathrm{d}t}{t} \right)^{1/q} \tag{2.19}$$

where $s < m \in \mathbb{N}$. This is the counterpart of (1.24) for bounded Lipschitz domains Ω. There is also an appropriate extension of this assertion to

$$B^s_{pq}(\Omega), \quad 0 < p, q \le \infty, \ s > n\big(\tfrac{1}{p} - 1\big)_+. \tag{2.20}$$

Details and references may be found in [T06], Theorem 1.118, Remark 1.119, pp. 74–76.

2.1.2 Wavelet systems and sequence spaces

Recall that (arbitrary) open sets Ω in $\mathbb{R}^n$ are called domains. We always assume that $\Omega \ne \mathbb{R}^n$ (hence $\Gamma = \partial\Omega \ne \emptyset$). If Γ^1 and Γ^2 are two sets in $\mathbb{R}^n$ then

$$\operatorname{dist}(\Gamma^1, \Gamma^2) = \inf \big\{ |x^1 - x^2| : \ x^1 \in \Gamma^1, \ x^2 \in \Gamma^2 \big\}. \tag{2.21}$$

We rely on the well-known *Whitney decomposition* as it may be found in [Ste70], Theorem 3, p. 16; Theorem 1, p. 167, adapted to our needs. Let

$$Q^0_{lr} \subset Q^1_{lr}, \quad l \in \mathbb{N}_0; \ r = 1, 2, \dots, \tag{2.22}$$

be concentric (open) cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-l} m^r$ for some $m^r \in \mathbb{Z}^n$ with the respective side-lengths 2^{-l} and 2^{-l+1} such that for suitable pairwise disjoint cubes Q^0_{lr},

$$\Omega = \bigcup_{l,r} \overline{Q^0_{lr}} \quad \text{and} \quad \operatorname{dist}(Q^1_{lr}, \Gamma) \sim 2^{-l} \quad \text{if } l \in \mathbb{N}, \tag{2.23}$$

complemented by $\operatorname{dist}(Q^1_{0r}, \Gamma) \ge c$ for some $c > 0$. For adjacent cubes Q^0_{lr}, $Q^0_{l'r'}$ one may assume that $|l - l'| \le 1$. We used Whitney decompositions of this type in [T06], Section 4.2.3 (based on [Tri07a]) and in [Tri07b] to construct so-called wavelet para-bases in some B-spaces and F-spaces in Ω. The elements of these para-bases preserve the structure of the $\mathbb{R}^n$-wavelets according to (1.91). But now we wish to convert these para-bases into bases. But this requires some additional multi-resolution arguments and local orthonormalisation. As a consequence one is losing the one-to-one relation between Whitney cubes and the index sets for the related bases. This suggests to give priority to the bases and to describe their properties and relations to wavelets (localisation, smoothness, cancellation) in qualitative terms. For this purpose we introduce the counterparts of the wavelet systems (1.91) and the sequence spaces in Definition 1.18 in $\mathbb{R}^n$ now based on the following *approximate lattices* $\mathbb{Z}_\Omega$ in Ω.

Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \ne \mathbb{R}^n$ (hence $\Gamma = \partial\Omega \ne \emptyset$) and let for some positive numbers c_1, c_2, c_3,

$$\mathbb{Z}_\Omega = \bigl\{ x^j_r \in \Omega : \; j \in \mathbb{N}_0; \; r = 1, \ldots, N_j \bigr\} \tag{2.24}$$

where $N_j \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ such that

$$|x^j_r - x^j_{r'}| \ge c_1 \, 2^{-j}, \quad j \in \mathbb{N}_0, \; r \ne r', \tag{2.25}$$

and

$$\operatorname{dist}\Bigl(\bigcup_{r=1}^{N_j} B\bigl(x^j_r, c_2 \, 2^{-j}\bigr), \Gamma \Bigr) \ge c_3 \, 2^{-j}, \quad j \in \mathbb{N}_0. \tag{2.26}$$

Here $B(x, \varrho)$ is a ball centred at $x \in \mathbb{R}^n$ and of radius $\varrho > 0$. It is always assumed that the positive numbers c_1, c_2, c_3 are sufficiently small in dependence on Ω such that for any $j \in \mathbb{N}_0$ there are points x^j_r with (2.24)–(2.26) (as suggested by $N_j \in \overline{\mathbb{N}}$). Let for $j \in \mathbb{N}_0$ and $G \in G^0 = \{F, M\}^n$,

$$\Psi^{j,L}_{G,m}(x) = 2^{(j+L)n/2} \prod_{r=1}^{n} \psi_{G_r} \bigl(2^{j+L} x_r - m_r \bigr), \quad m \in \mathbb{Z}^n, \tag{2.27}$$

be as in (1.153), based on the dilated real compactly supported Daubechies wavelets (1.152),

$$\psi^L_F = \psi_F(2^L \cdot) \in C^u(\mathbb{R}), \quad \psi^L_M = \psi_M(2^L \cdot) \in C^u(\mathbb{R}), \quad u \in \mathbb{N}, \tag{2.28}$$

and (1.88), where we choose $L \in \mathbb{N}$ such that $\varepsilon > 0$ in

$$\operatorname{supp} \psi^L_F \subset (-\varepsilon, \varepsilon), \quad \operatorname{supp} \psi^L_M \subset (-\varepsilon, \varepsilon) \tag{2.29}$$

is small (as specified later on). Let $\overline{F} = \{F, \ldots, F\} \in \{F, M\}^n$.

Definition 2.4. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and let $\mathbb{Z}_\Omega$ be as in (2.24)–(2.26). Let $L \in \mathbb{N}$ and $u \in \mathbb{N}$ as in (2.27), (2.28), and (1.88). Let $K \in \mathbb{N}$, $D > 0$ and $c_4 > 0$. Then

$$\left\{\Phi_r^j : j \in \mathbb{N}_0;\ r = 1, \ldots, N_j\right\} \quad \text{where } N_j \in \bar{\mathbb{N}} \tag{2.30}$$

with

$$\operatorname{supp} \Phi_r^j \subset B\left(x_r^j, c_2\, 2^{-j}\right), \quad j \in \mathbb{N}_0, \tag{2.31}$$

$B(x_r^j, c_2 2^{-j})$ as in (2.26), is called a u-wavelet system (with respect to Ω) if it consists of the following three types of functions.

(i) Basic wavelets

$$\Phi_r^0 = \Psi_{G,m}^{0,L} \quad \text{for some } G \in \{F, M\}^n,\ m \in \mathbb{Z}^n. \tag{2.32}$$

(ii) Interior wavelets

$$\Phi_r^j = \Psi_{G,m}^{j,L}, \quad j \in \mathbb{N},\ \operatorname{dist}(x_r^j, \Gamma) \geq c_4\, 2^{-j}, \tag{2.33}$$

for some $G \in \{F, M\}^{n*}$, $m \in \mathbb{Z}^n$.

(iii) Boundary wavelets

$$\Phi_r^j = \sum_{|m-m'| \leq K} d_{m,m'}^j\, \Psi_{\bar{F},m'}^{j,L}, \quad j \in \mathbb{N},\ \operatorname{dist}(x_r^j, \Gamma) < c_4\, 2^{-j}, \tag{2.34}$$

for some $m = m(j, r) \in \mathbb{Z}^n$ and $d_{m,m'}^j \in \mathbb{R}$ with

$$\sum_{|m-m'| \leq K} |d_{m,m'}^j| \leq D \quad \text{and} \quad \operatorname{supp} \Psi_{\bar{F},m'}^{j,L} \subset B(x_r^j, c_2 2^{-j}). \tag{2.35}$$

Remark 2.5. By construction all wavelets Φ_r^j are real. Just as for wavelet bases in $\mathbb{R}^n$ and $\mathbb{T}^n$ only the number $u \in \mathbb{N}$ is of interest. All other numbers L, K, D and the constants $c_1, \ldots, c_4$ are technical ingredients. It comes out that they depend on u, but not on Ω. This may justify the above notation. It is our first aim to show that for any $u \in \mathbb{N}$ there are u-wavelet systems which are also orthonormal bases in $L_2(\Omega)$. As in case of $\mathbb{R}^n$ and $\mathbb{T}^n$ one can prove afterwards that the complete orthonormal u-wavelet systems are also unconditional bases in some function spaces of B-type and F-type. For this purpose one needs the counterparts of the sequence spaces in Definitions 1.18 and 1.32. Let χ_{jr} be the characteristic function of the ball $B(x_r^j, c_2 2^{-j})$ in (2.31).

Definition 2.6. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and let $\mathbb{Z}_\Omega$ be as in (2.24)–(2.26). Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. Then $b_{pq}^s(\mathbb{Z}_\Omega)$ is the collection of all sequences

$$\lambda = \left\{\lambda_r^j \in \mathbb{C} : j \in \mathbb{N}_0;\ r = 1, \ldots, N_j\right\}, \quad N_j \in \bar{\mathbb{N}}, \tag{2.36}$$

such that

$$\|\lambda \,|b^s_{pq}(\mathbb{Z}_\Omega)\| = \Big(\sum_{j=0}^\infty 2^{j(s-\frac{n}{p})q} \Big(\sum_{r=1}^{N_j} |\lambda^j_r|^p \Big)^{q/p} \Big)^{1/q} < \infty, \tag{2.37}$$

and $f^s_{pq}(\mathbb{Z}_\Omega)$ is the collection of all sequences (2.36) such that

$$\|\lambda \,|f^s_{pq}(\mathbb{Z}_\Omega)\| = \Big\| \Big(\sum_{j=0}^\infty \sum_{r=1}^{N_j} 2^{jsq} \,|\lambda^j_r \,\chi_{jr}(\cdot)|^q \Big)^{1/q} |L_p(\Omega) \Big\| < \infty \tag{2.38}$$

(obviously modified if $p = \infty$ and/or $q = \infty$).

Remark 2.7. There is no need to bother about the seeming ambiguity about the admitted distributions of the points $x^j_r \in \mathbb{Z}_\Omega$, the radius $c_2\, 2^{-j}$ of the balls in (2.31), or the replacement of the balls by suitable cubes in connection with the characteristic functions in (2.38). All this can be justified by the related observations in [T06], Section 1.5.3 (at the expense of equivalent quasi-norms, $p < \infty$). Furthermore,

$$f^s_{pp}(\mathbb{Z}_\Omega) = b^s_{pp}(\mathbb{Z}_\Omega), \quad s \in \mathbb{R},\ 0 < p \le \infty.$$

Remark 2.8. The wavelet expansions according to Theorem 1.20 for spaces on $\mathbb{R}^n$ and their periodic counterparts in Theorem 1.36 rely on the lattices $2^{-j}\mathbb{Z}^n$. We extended this theory in [T06], Section 4.2 (based on [Tri07a]) and [Tri07b] to domains in $\mathbb{R}^n$ preserving essentially this lattice structure. We obtained distinguished frames, called para-bases, where the main part, being a basis in its closed hull, comes from the above basic wavelets (2.32) and the above interior wavelets (2.33). The remainder part preserves the lattice structure but spoils the property of being a basis. Now we wish to convert this remainder part into a basis in its closed hull as indicated at the beginning of this Section 2.1.2. This is reflected by the boundary wavelets in (2.34). However it seems to be reasonable to abandon the lattice structure. This may justify the introduction of u-wavelet systems according to Definition 2.4 now based on the *approximate lattices* in $\mathbb{Z}_\Omega$.

2.2 Homogeneity and refined localisation spaces

2.2.1 Homogeneity

Recall that this book may be considered as the continuation of [T06] (and [T01]). We rely on the results obtained there. We proved in [T01], Theorem 5.14, pp. 60–61, for bounded C^∞ domains and in [T06], Proposition 4.20, pp. 208–09, for bounded Lipschitz domains that some spaces $\widetilde{F}^s_{pq}(\Omega)$ according to Definition 2.1 (ii) have the refined localisation property which will be the subject of Section 2.2.3. This is closely connected with local homogeneity assertions for some spaces F^s_{pq} as proved first in [T01], Corollary 5.16, p. 66. This had been extended in [CLT07] to some spaces B^s_{pq}. Now we return to this subject in greater generality both for its own sake but also as a crucial ingredient for the study of refined localisation spaces below.

Let σ_p and σ_{pq} be the same numbers as in (1.32) and let either

$$U_\lambda = \{x \in \mathbb{R}^n : |x| < \lambda\} \quad \text{or} \quad U_\lambda = \{x \in \mathbb{R}^n : |x_r| < \lambda\} \tag{2.39}$$

(balls or cubes) where $0 < \lambda < \infty$. Let $A^s_{pq}(\Omega)$ and $\tilde{A}^s_{pq}(\Omega)$ be the spaces as introduced in Definition 2.1.

Definition 2.9. Let U_λ be either the balls or the cubes according to (2.39). Then

$$\bar{F}^s_{pq}(U_\lambda) = \begin{cases} \tilde{F}^s_{pq}(U_\lambda) & \text{if } 0 < p < \infty, 0 < q \le \infty, s > \sigma_{pq}, \\ F^0_{pq}(U_\lambda) & \text{if } 1 < p < \infty, 1 \le q < \infty, s = 0, \\ F^s_{pq}(U_\lambda) & \text{if } 0 < p < \infty, 0 < q \le \infty, s < 0, \end{cases} \tag{2.40}$$

and

$$\bar{B}^s_{pq}(U_\lambda) = \begin{cases} \tilde{B}^s_{pq}(U_\lambda) & \text{if } 0 < p \le \infty, 0 < q \le \infty, s > \sigma_p, \\ B^0_{pq}(U_\lambda) & \text{if } 1 < p < \infty, 0 < q \le \infty, s = 0, \\ B^s_{pq}(U_\lambda) & \text{if } 0 < p \le \infty, 0 < q \le \infty, s < 0. \end{cases} \tag{2.41}$$

Remark 2.10. One observation about the above restrictions for the domains and the parameters seems to be in order. In Section 2.2.3 we introduce the refined localisation spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ for arbitrary domains Ω in $\mathbb{R}^n$ for the same parameters as in (2.40). It turns out that

$$F^{s,\mathrm{rloc}}_{pq}(U_\lambda) = \bar{F}^s_{pq}(U_\lambda). \tag{2.42}$$

This equality with Ω in place of U_λ is not true in general for arbitrary domains, but for so-called E-thick domains which include balls and cubes. We refer to Theorem 3.28. This may justify the above definition including the restrictions of the parameters.

Theorem 2.11. (Homogeneity property) *Let U_λ be either the balls or the cubes according to (2.39) and let $\bar{A}^s_{pq}(U_\lambda)$ with $A = B$ or $A = F$ be one of the spaces from Definition 2.9. Then*

$$\|f(\lambda\,\cdot)\,|\bar{A}^s_{pq}(U_1)\| \sim \lambda^{s-\frac{n}{p}}\,\|f\,|\bar{A}^s_{pq}(U_\lambda)\| \tag{2.43}$$

where the equivalence constants are independent of λ with $0 < \lambda \le 1$ and of $f \in \bar{A}^s_{pq}(U_\lambda)$.

Remark 2.12. The case

$$\|f(\lambda\,\cdot)\,|\bar{F}^s_{pq}(U_1)\| \sim \lambda^{s-\frac{n}{p}}\,\|f\,|\bar{F}^s_{pq}(U_\lambda)\|, \quad 0 < \lambda \le 1, \tag{2.44}$$

with

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s > \sigma_{pq}, \tag{2.45}$$

(where $q = \infty$ if $p = \infty$ and $\bar{F}^s_{\infty\infty} = \bar{B}^s_{\infty\infty}$) is covered by [T01], Corollary 5.16, p. 66. By Definition 2.1 one can reformulate (2.44) as

$$\| f(\lambda \cdot) \,|F^s_{pq}(\mathbb{R}^n)\| \sim \lambda^{s-\frac{n}{p}} \, \| f \,|F^s_{pq}(\mathbb{R}^n)\|, \quad \operatorname{supp} f \subset \bar{U}_\lambda, \tag{2.46}$$

where again $0 < \lambda \leq 1$ and s, p, q are as in (2.45). This local homogeneity is a cornerstone of what follows. It will be used to prove the existence of wavelet bases in refined localisation spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ with (2.45) for arbitrary domains Ω in $\mathbb{R}^n$. In Section 3.2 we consider wavelet bases in the other spaces covered by Definition 2.9. Afterwards we prove in Section 3.3.2 the homogeneity assertion (2.43) for these spaces. A direct proof of (2.43) (without using wavelet expansions) for the spaces

$$\bar{B}^s_{pq}(U_\lambda) \quad \text{with } 0 < p, q \leq \infty, \; s > \sigma_p, \tag{2.47}$$

has been given recently in [CLT07].

2.2.2 Pointwise multipliers

In [T01], Section 5.17, p. 67, we used the homogeneity property (2.46) with (2.45) for some adapted multiplier assertions. This can now be extended to all spaces covered by Theorem 2.11. Let $C^\varkappa(\mathbb{R}^n)$ with $\varkappa \geq 0$ be the usual spaces, normed by

$$\|g \,|C^\varkappa(\mathbb{R}^n)\| = \sum_{|\alpha| \leq \varkappa} \sup_{x \in \mathbb{R}^n} |D^\alpha g(x)| \quad \text{if } \varkappa \in \mathbb{N}_0 \tag{2.48}$$

and

$$\|g \,|C^\varkappa(\mathbb{R}^n)\| = \|g \,|C^{[\varkappa]}(\mathbb{R}^n)\| + \sum_{|\alpha| = [\varkappa]} \sup \frac{|D^\alpha g(x) - D^\alpha g(y)|}{|x-y|^{\{\varkappa\}}} \tag{2.49}$$

if $0 < \varkappa = [\varkappa] + \{\varkappa\}$ with $[\varkappa] \in \mathbb{N}_0$ and $0 < \{\varkappa\} < 1$, where the latter supremum is taken over all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ with $0 < |x-y| \leq 1$. Let $s \in \mathbb{R}$, $0 < p \leq \infty$ ($p < \infty$ for the F-spaces), $0 < q \leq \infty$, and $\varkappa > \max(s, \sigma_p - s)$. According to [T92], Corollary, p. 205, there is a positive number c such that

$$\|gf \,|A^s_{pq}(\mathbb{R}^n)\| \leq c \, \|g \,|C^\varkappa(\mathbb{R}^n)\| \cdot \|f \,|A^s_{pq}(\mathbb{R}^n)\| \tag{2.50}$$

for all $g \in C^\varkappa(\mathbb{R}^n)$ and all $f \in A^s_{pq}(\mathbb{R}^n)$ where again $A = B$ or $A = F$ (pointwise multiplier property). This assertion is sharp in the following sense: For any $0 < \varkappa < \max(s, \sigma_p - s)$ there are functions $g \in C^\varkappa(\mathbb{R}^n)$ which are not pointwise multipliers for $A^s_{pq}(\mathbb{R}^n)$, [T92], Remark 3, p. 206, with a reference to [T83], Corollary, p. 143. Otherwise pointwise multipliers in the spaces $A^s_{pq}(\mathbb{R}^n)$ are not the subject of this book. In addition to [T83], [T92] one finds more recent results and related references in [T06], Section 2.3. We are only interested in a local assertion combining Theorem 2.11 with (2.50).

Theorem 2.13. *Let* $\overline{A}^s_{pq}(U_\lambda)$ *be one of the spaces according to Definition 2.9 with* $0 < \lambda \le 1$. *Let* $\varkappa > \max(s, \sigma_p - s)$. *Then*

$$\|gf \,|\overline{A}^s_{pq}(U_\lambda)\| \le c\, \|g(\lambda\,\cdot)\,|C^\varkappa(\mathbb{R}^n)\| \cdot \|f\,|\overline{A}^s_{pq}(U_\lambda)\| \tag{2.51}$$

where c *is a positive constant which is independent of* $f \in \overline{A}^s_{pq}(U_\lambda)$, λ *with* $0 < \lambda \le 1$, *and*

$$g \in C^\varkappa(\mathbb{R}^n) \quad \text{with } \operatorname{supp} g \subset U_{2\lambda}. \tag{2.52}$$

Proof. By (2.43) and (2.50) we have

$$\begin{aligned} \|gf \,|\overline{A}^s_{pq}(U_\lambda)\| &\sim \lambda^{\frac{n}{p}-s}\, \|g(\lambda\,\cdot)f(\lambda\cdot)\,|\overline{A}^s_{pq}(U_1)\| \\ &\le c\,\lambda^{\frac{n}{p}-s}\, \|g(\lambda\,\cdot)\,|C^\varkappa(\mathbb{R}^n)\| \cdot \|f(\lambda\,\cdot)\,|\overline{A}^s_{pq}(U_1)\|, \end{aligned} \tag{2.53}$$

where we used some standard properties of function spaces in unit balls or unit cubes such as extension and multiplication with suitable cut-off functions. Re-transformation of (2.53) gives (2.51). □

2.2.3 Refined localisation spaces

Let Ω be an arbitrary domain in $\mathbb{R}^n$ and let Q^0_{lr}, Q^1_{lr} be the Whitney cubes according to (2.22), (2.23) and the other properties described there. Let $\varrho = \{\varrho_{lr}\}$ be a related resolution of unity with

$$\operatorname{supp} \varrho_{lr} \subset Q^1_{lr}, \quad |D^\gamma \varrho_{lr}(x)| \le c_\gamma\, 2^{l|\gamma|}, \quad x \in \Omega,\ \gamma \in \mathbb{N}^n_0, \tag{2.54}$$

for some $c_\gamma > 0$ and

$$\sum_{l=0}^\infty \sum_r \varrho_{lr}(x) = 1 \quad \text{if } x \in \Omega. \tag{2.55}$$

Let $F^s_{\infty\infty} = B^s_{\infty\infty}$. Let σ_{pq} be the same number as in (1.32).

Definition 2.14. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \ne \mathbb{R}^n$. Let $\varrho = \{\varrho_{lr}\}$ be the above resolution of unity. Then

$$F^{s,\mathrm{rloc}}_{pq}(\Omega) = \big\{ f \in D'(\Omega) :\ \|f\,|F^{s,\mathrm{rloc}}_{pq}(\Omega)\|_\varrho < \infty \big\} \tag{2.56}$$

with

$$\|f\,|F^{s,\mathrm{rloc}}_{pq}(\Omega)\|_\varrho = \Big(\sum_{l=0}^\infty \sum_r \|\varrho_{lr} f\,|F^s_{pq}(\mathbb{R}^n)\|^p \Big)^{1/p} \tag{2.57}$$

when

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s > \sigma_{pq}, \tag{2.58}$$

($q = \infty$ if $p = \infty$) and

$$\|f \,| F^{s,\mathrm{rloc}}_{pq}(\Omega)\|_\varrho = \Big(\sum_{l=0}^{\infty} \sum_r \|\varrho_{lr} f \,| F^s_{pq}(Q^1_{lr})\|^p \Big)^{1/p} \tag{2.59}$$

when

$$\left. \begin{aligned} &\text{either } 1 < p < \infty,\ 1 \le q < \infty,\ s = 0, \\ &\text{or } 0 < p \le \infty,\ 0 < q \le \infty,\ s < 0 \end{aligned} \right\} \tag{2.60}$$

(again $q = \infty$ if $p = \infty$).

Remark 2.15. The parameters p, q, s in (2.58), (2.60) are the same as in (2.40) complemented by $p = q = \infty$, covered by (2.41). Obviously, $\varrho_{lr} f$ with $f \in D'(\Omega)$ is extended outside of Q^1_{lr} by zero. This is justified by (2.54). Then both (2.57) and (2.59) make sense. In case of (2.57), (2.58) one can replace $F^s_{pq}(\mathbb{R}^n)$ by $\widetilde{F}^s_{pq}(Q^1_{lr})$, hence

$$\|f \,| F^{s,\mathrm{rloc}}_{pq}(\Omega)\|_\varrho = \Big(\sum_{l=0}^{\infty} \sum_r \|\varrho_{lr} f \,| \widetilde{F}^s_{pq}(Q^1_{lr})\|^p \Big)^{1/p}. \tag{2.61}$$

This follows from Definition 2.1 and Remark 2.2. Since $\operatorname{supp} \varrho_{lr} f \subset Q^1_{lr}$ one may ask whether one can replace $F^s_{pq}(Q^1_{lr})$ in (2.59), (2.60) by $F^s_{pq}(\mathbb{R}^n)$. But this is not the case. We return to this point in Remark 2.22 below.

Theorem 2.16. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Then the spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ according to Definition 2.14 are quasi-Banach spaces. They are independent of $\varrho = \{\varrho_{lr}\}$ (equivalent quasi-norms).*

Proof. Recall that (2.57) can be replaced by (2.61). Then the independence of the spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ of $\varrho = \{\varrho_{lr}\}$ follows from the pointwise multiplier assertion in Theorem 2.13 applied to $g = \varrho_{lr}$ with (2.54) and a possible second resolution of unity. Afterwards one obtains by standard arguments that $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ is a quasi-Banach space. □

Remark 2.17. We used only the pointwise multiplier property according to Theorem 2.13. In other words, if $\overline{A}^s_{pq}$ is one of the spaces covered by Theorem 2.13, Definition 2.9, then the above arguments can also be applied to spaces quasi-normed by

$$\Big(\sum_{l=0}^{\infty} \sum_r \|\varrho_{lr} f \,| \overline{A}^s_{pq}(Q^1_{lr})\|^v \Big)^{1/v}, \quad 0 < v \le \infty. \tag{2.62}$$

But it is not clear whether these more general spaces are of any interest. In case of the spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ as introduced in Definition 2.14 the situation is somewhat different. The first step was taken in [T01], Theorem 5.14, where we proved for bounded C^∞ domains Ω in $\mathbb{R}^n$ that

$$F^{s,\mathrm{rloc}}_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega), \quad 0 < p, q \le \infty,\ s > \sigma_{pq}, \tag{2.63}$$

($q = \infty$ if $p = \infty$). Here $\widetilde{F}^s_{pq}(\Omega)$ are the same spaces as introduced in Definition 2.1 and commented in Remark 2.2. We called this the *refined localisation property* of the spaces $\widetilde{F}^s_{pq}(\Omega)$. We extended this assertion in [T06], Proposition 4.20, Remark 4.21, to bounded Lipschitz domains. Now we took the refined localisation property in Definition 2.14 as the motivation to introduce the spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ with (2.57), (2.58) for arbitrary domains. On the one hand we prove (2.63) later on in Proposition 3.10 for so-called E-thick domains. On the other hand, (2.63) cannot be expected for arbitrary domains, even if one assumes that $|\Gamma| = 0$ with $\Gamma = \partial\Omega$ such that one has at least (2.9), (2.10). For example, if

$$\Omega \neq \mathbb{R}^n, \quad \bar{\Omega} = \mathbb{R}^n \quad \text{and} \quad |\Gamma| = 0, \tag{2.64}$$

then $\widetilde{F}^s_{pq}(\Omega) = F^s_{pq}(\mathbb{R}^n)$. However if $p < \infty$, $q < \infty$, then $D(\Omega)$ is dense in $F^{s,\mathrm{rloc}}_{pq}(\Omega)$. This contradicts (2.63) if $s > n/p$ (then $F^s_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$). Later on we prove in Theorem 3.28 and Remark 3.29 for E-thick domains Ω that

$$F^{s,\mathrm{rloc}}_{pq}(\Omega) = F^s_{pq}(\Omega) \quad \text{if } 0 < p,q \le \infty,\ s < 0, \tag{2.65}$$

($q = \infty$ if $p = \infty$). This justifies the definition of the spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ with (2.59), (2.60).

First we characterise $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ with $s > \sigma_{pq}$ in terms of the ball means of differences in $\mathbb{R}^n$,

$$d^M_{t,u} f(x) = \left(t^{-n} \int_{|h|\le t} |\Delta^M_h f(x)|^u \, \mathrm{d}h \right)^{1/u}, \quad x \in \mathbb{R}^n,\ t > 0, \tag{2.66}$$

where $0 < u \le \infty$ (usual modification if $u = \infty$) and where Δ^M_h are the usual differences according to (1.21). By [T92], Theorem 3.5.3, p. 194, one can characterise the spaces $F^s_{pq}(\mathbb{R}^n)$, $s > \sigma_{pq}$, in terms of these ball means. We refer also to [T06], Section 1.11.9. Let Ω be again an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$, hence $\Gamma = \partial\Omega \neq \emptyset$, and let

$$\delta(x) = \min\big(1, \mathrm{dist}(x,\Gamma)\big), \quad x \in \Omega. \tag{2.67}$$

For $M \in \mathbb{N}$ let $\varkappa$ with $0 < \varkappa < 1$ and $c > 0$ be numbers such that

$$B(x, Mt) \subset \Omega, \quad \mathrm{dist}(B(x,Mt),\Gamma) \ge c\,\delta(x), \tag{2.68}$$

for all $x \in \Omega$ and all $0 < t \le \varkappa\,\delta(x)$. Recall that $B(x,r)$ stands for a ball centred at $x \in \mathbb{R}^n$ and of radius $r > 0$.

Theorem 2.18. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and let $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ be the spaces according to Definition 2.14 with*

$$0 < p \le \infty, \quad 0 < q \le \infty \ (\textit{with } q = \infty \textit{ if } p = \infty), \quad s > \sigma_{pq}. \tag{2.69}$$

(i) *Let*

$$\max(1,p) < v \le \infty, \quad s - \frac{n}{p} > -\frac{n}{v} \tag{2.70}$$

(*which means* $v = \infty$ *if* $p = \infty$). *Then*

$$F^{s,\mathrm{rloc}}_{pq}(\Omega) \hookrightarrow L_v(\Omega) \tag{2.71}$$

(*continuous embedding*).

(ii) *Let* $0 < u < \min(1,p,q)$ *and* $s < M \in \mathbb{N}$ *in* (2.66). *Let* $\varkappa$ *be as above. Then* $f \in L_v(\Omega)$ *with* v *as in* (2.70) *belongs to* $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ *if, and only if,*

$$\Big\| \Big(\int_0^{\varkappa\delta(\cdot)} t^{-sq}\, d^M_{t,u} f(\cdot)^q \, \frac{\mathrm{d}t}{t}\Big)^{1/q} |L_p(\Omega)\Big\| + \|\delta^{-s}(\cdot) f\,|L_p(\Omega)\| < \infty \tag{2.72}$$

(*equivalent quasi-norms*).

Proof. Recall the well-known continuous embedding

$$F^s_{pq}(\mathbb{R}^n) \hookrightarrow L_v(\mathbb{R}^n), \quad s - \tfrac{n}{p} > -\tfrac{n}{v}. \tag{2.73}$$

Then (2.71) follows from (2.57), $p \le v$, and an obvious counterpart of (2.57) for $L_v(\Omega)$. Part (ii) is essentially covered by [T01], Corollary 5.15, p. 66, and the underlying proof. □

Remark 2.19. The rather long and complicated proof of (2.63) for bounded C^∞ domains in $\mathbb{R}^n$ was just based on characterisations of $F^s_{pq}(\mathbb{R}^n)$ in terms of ball means and one obtained equivalent quasi-norms of type (2.72) as a by-product. Now we defined $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ via refined localisation. Then the corresponding arguments in [T01] can be used without changes. As indicated above we prove later on (2.63) for the larger class of E-thick domains by means of wavelet bases.

Corollary 2.20. *Let* Ω *be an arbitrary domain in* $\mathbb{R}^n$ *with* $\Omega \ne \mathbb{R}^n$ *and let*

$$W^{k,\mathrm{rloc}}_p(\Omega) = F^{k,\mathrm{rloc}}_{p,2}(\Omega), \quad k \in \mathbb{N},\ 1 < p < \infty. \tag{2.74}$$

Then $W^{k,\mathrm{rloc}}_p(\Omega)$ *is the collection of all* $f \in L_p(\Omega)$ *such that*

$$\begin{aligned} \|f\,|W^{k,\mathrm{rloc}}_p(\Omega)\| &= \sum_{|\alpha|\le k} \|\delta^{-k+|\alpha|} D^\alpha f\,|L_p(\Omega)\| \\ &\sim \sum_{|\alpha|=k} \|D^\alpha f\,|L_p(\Omega)\| + \|\delta^{-k} f\,|L_p(\Omega)\| \end{aligned} \tag{2.75}$$

is finite (*equivalent norms*).

Proof. This follows by elementary reasoning from (2.57) and the well-known Paley–Littlewood assertion

$$F^k_{p,2}(\mathbb{R}^n) = W^k_p(\mathbb{R}^n), \quad k \in \mathbb{N},\ 1 < p < \infty. \tag{2.76}$$

□

Remark 2.21. Some thirty years ago we dealt in [T78], Chapter 3, with problems of this type for (fractional) Sobolev spaces and Besov spaces.

Remark 2.22. Finally we return to the question why one cannot replace $F^s_{pq}(Q^1_{lr})$ in (2.59) by $F^s_{pq}(\mathbb{R}^n)$ although $\varrho_{rl} f$ has compact support in Q^1_{lr} (as we did erroneously in [T01], Section 5.20, p. 68). This can be reduced to a corresponding homogeneity problem (2.43) for spaces A^s_{pq} with $s < 0$. Let

$$\psi(x) > 0 \text{ if } x \in U_1 = \{y : |y| < 1\} \quad \text{and} \quad \operatorname{supp}\psi = \overline{U_1}, \tag{2.77}$$

be a C^∞ bump function in $\mathbb{R}^n$. Let

$$\psi^{s,p}_j(x) = 2^{-j(s-\frac{n}{p})}\,\psi(2^j x), \quad j \in \mathbb{N}_0. \tag{2.78}$$

With $U^j = U_{2^{-j}}$ one obtains by (2.43) for $0 < p, q \le \infty$,

$$\|\psi^{s,p}_j \,|B^s_{pq}(\mathbb{R}^n)\| \sim \|\psi\,|B^s_{pq}(\mathbb{R}^n)\| \sim 1 \quad \text{if } s > \sigma_p \tag{2.79}$$

and

$$\|\psi^{s,p}_j \,|B^s_{pq}(U^j)\| \sim \|\psi\,|B^s_{pq}(\mathbb{R}^n)\| \sim 1 \quad \text{if } s < 0. \tag{2.80}$$

We used the B-counterpart of (2.46). If $s < 0$ then one can rely on equivalent quasi-norms for the spaces $B^s_{pq}(\mathbb{R}^n)$ in terms of local means with non-negative kernels as may be found in [T06], Corollary 1.12, p. 11. One finds (after some calculations) that

$$\|\psi^{s,p}_j \,|B^s_{pq}(\mathbb{R}^n)\| \sim 2^{j|s|}, \quad j \in \mathbb{N}_0,\ s < 0. \tag{2.81}$$

To provide a better understanding of the behaviour of $\psi^{s,p}_j$ in $B^s_{pq}(\mathbb{R}^n)$ and in $B^s_{pq}(U^j)$ we first remark that (2.78) are normalised atoms in $B^s_{pq}(\mathbb{R}^n)$ if $s > \sigma_p$. This is in good agreement with (2.79). If $s < 0$ then one needs moment conditions for the atoms. To obtain (2.80) one can complement $\psi^{s,p}_j$ in $U^{2j}\backslash U^j$ by suitable functions generating the needed moment conditions. This explains (2.80). However in case of (2.81) there is no such possibility and the outcome shows that $\psi^{s,p}_j$ are no longer normalised atoms in $B^s_{pq}(\mathbb{R}^n)$ with $s < 0$.

2.3 Wavelet para-bases

2.3.1 Some preparations

Recall that this book should be considered as the continuation of [T06] and that we rely on the assertions obtained there. In [T06], Section 4.2, we dealt with wavelet para-bases in some spaces $A^s_{pq}(\Omega)$ and $\tilde{A}^s_{pq}(\Omega)$ according to Definition 2.1 in bounded Lipschitz domains in $\mathbb{R}^n$ (based on [Tri07a]). The starting point was the refined localisation property

$$\widetilde{F}^s_{pq}(\Omega) = F^{s,\mathrm{rloc}}_{pq}(\Omega), \quad 1 < p, q \le \infty, \; s > 0, \tag{2.82}$$

(with $q = \infty$ if $p = \infty$), [T06], Proposition 4.20, p. 208, where $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ has the same meaning as in Definition 2.14. But afterwards we needed only this refined localisation property combined with the homogeneity (2.43) for the spaces in the first line of (2.40), hence (2.44), (2.45). This makes clear that there is no problem to extend the considerations in [T06], Section 4.2, to the refined localisation spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ according to (2.56)–(2.58). Nevertheless we give a careful description which will be the starting point for the construction of wavelet bases according to Definition 2.4 in spaces of this type.

Again let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ (hence $\Gamma = \partial\Omega \neq \emptyset$). We modify the *Whitney decomposition* (2.22), (2.23) as follows. Let

$$Q^0_{lr} \subset Q^1_{lr} \subset Q^2_{lr} \subset Q_{lr}, \quad l \in \mathbb{N}_0; \; r = 1, 2, \ldots, \tag{2.83}$$

be concentric open cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates centred at $2^{-l}m^r$ for some $m^r \in \mathbb{Z}^n$ with the respective side-lengths $2^{-l}, 5 \cdot 2^{-l-2}, 6 \cdot 2^{-l-2}$, 2^{-l+1}. According to the Whitney decomposition there are disjoint cubes Q^0_{lr} of this type such that

$$\Omega = \bigcup_{l,r} \overline{Q^0_{lr}} \quad \text{and} \quad \operatorname{dist}(Q_{lr}, \Gamma) \sim 2^{-l} \tag{2.84}$$

if $l \in \mathbb{N}$ and $r = 1, 2, \ldots$, complemented by $\operatorname{dist}(Q_{0r}, \Gamma) \ge c$ for some $c > 0$. We may assume that $|l - l'| \le 1$ for any two admitted cubes Q^1_{lr}, $Q^1_{l'r'}$, having a non-empty intersection. We follow the construction of wavelet para-bases given in [T06], Section 4.2.4, now for arbitrary domains. We adapt the real wavelets in (2.27), (2.28), based on (1.87), (1.88), by

$$\Psi^j_{G,m}(x) = 2^{(j+L)n/2} \prod_{a=1}^n \psi_{G_a}\left(2^{j+L}x_a - m_a\right), \quad G \in \{F, M\}^n, \; m \in \mathbb{Z}^n, \tag{2.85}$$

where $L \in \mathbb{N}_0$ is fixed such that

$$\operatorname{supp} \Psi^j_{G,m} \subset Q_{lr} \quad \text{if } 2^{-j-L}m \in Q^2_{lr} \text{ for } l \in \mathbb{N}_0 \text{ and } j \ge l, \tag{2.86}$$

and

$$2^{-L-j}m \in Q^2_{lr} \quad \text{if } Q^1_{lr} \cap \operatorname{supp} \Psi^j_{G,m} \neq \emptyset \text{ for } l \in \mathbb{N}_0 \text{ and } j \ge l, \tag{2.87}$$

for admitted cubes according to (2.83), (2.84). The number $L \in \mathbb{N}_0$ depends on ψ_F, ψ_M and in particular on u in (1.87), (1.88), but it has nothing to do with Ω and is assumed to be fixed. It specifies ε in (2.29). Let $\{F, M\}^n$ and $\{F, M\}^{n*}$ be as in (1.89), (1.90). As said we follow [T06], Section 4.2.4, where one finds further details and explanations. We will be brief. Let for $j \in \mathbb{N}_0$,

$$S_j^{\Omega,1} = \{F, M\}^{n*} \times \{m \in \mathbb{Z}^n : 2^{-j-L} m \in Q^2_{lr} \text{ for some } l < j, \text{ some } r\} \tag{2.88}$$

be the *main index set* (with $S_0^{\Omega,1} = \emptyset$) and

$$S_j^{\Omega,2} = \{F, M\}^n \times \{m \in \mathbb{Z}^n : 2^{-j-L} m \in Q^2_{jr} \text{ for some } r\} \setminus S_j^{\Omega,1} \tag{2.89}$$

be the *residual index set*. Let

$$S^\Omega = S^{\Omega,1} \cup S^{\Omega,2}, \quad S^{\Omega,1} = \bigcup_{j=1}^\infty S_j^{\Omega,1}, \quad S^{\Omega,2} = \bigcup_{j=0}^\infty S_j^{\Omega,2}, \tag{2.90}$$

and let

$$\Psi^{1,\Omega} = \left\{ \Psi^j_{G,m} : (j, G, m) \in S^{\Omega,1} \right\} \tag{2.91}$$

be the *main wavelet system* and

$$\Psi^{2,\Omega} = \left\{ \Psi^j_{G,m} : (j, G, m) \in S^{\Omega,2} \right\} \tag{2.92}$$

be the *residual wavelet system* where $\Psi^j_{G,m}$ are given by (2.85)–(2.87). We discuss briefly the outcome. The arguments in [T06], Section 4.2.4, are based on the refined localisation property (2.82), the homogeneity according to Theorem 2.11 and the reduction to wavelet expansions in $\mathbb{R}^n$ as described in Theorem 1.20 for the spaces $A^s_{pq}(\mathbb{R}^n)$ and in Section 1.2.1 for $L_2(\mathbb{R}^n)$. Having these ingredients the quality of the domain Ω is irrelevant. This will be specified in Section 2.3.2 for the spaces $F^{s,\text{rloc}}_{pq}(\Omega)$. But this applies also to $L_p(\Omega)$ with $1 < p < \infty$. One has in particular

$$L_2(\Omega) = L_2^{(1)}(\Omega) \oplus L_2^{(2)}(\Omega), \tag{2.93}$$

with

$$L_2^{(1)}(\Omega) = \operatorname{span} \Psi^{1,\Omega} \quad \text{and} \quad L_2^{(2)}(\Omega) = \operatorname{span} \Psi^{2,\Omega}. \tag{2.94}$$

By construction, $\Psi^{1,\Omega}$ is an orthonormal basis on $L_2^{(1)}(\Omega)$. Any element of the main wavelet system $\Psi^{1,\Omega}$ is orthogonal to any element of the residual wavelet system. The pairwise scalar products of the elements of $\Psi^{2,\Omega}$ generate a band-limited matrix caused by the mild overlapping of the supports of elements belonging to adjacent cubes. In any case, $\Psi^{2,\Omega}$ is locally finite and the cardinal number of elements of $\Psi^{2,\Omega}$ with supports intersecting a given compact subset of Ω is finite. It is one of our main goals to find u-wavelet systems according to Definition 2.4 which are orthonormal bases in $L_2(\Omega)$. Then we identify the interior wavelets in (2.33) with $\Psi^{1,\Omega}$ and convert $\Psi^{2,\Omega}$ by a specific orthonormalisation procedure into the basic and boundary wavelets (2.32), (2.35). But this will be shifted to Section 2.4. First we extend the theory of wavelet para-bases for the spaces $\widetilde{F}^s_{pq}(\Omega)$ in bounded Lipschitz domains Ω in $\mathbb{R}^n$ according to (2.82) to more general spaces. We follow at least partly [Tri07b].

2.3.2 Wavelet para-bases in $F^{s,\mathrm{rloc}}_{pq}(\Omega)$

Let Ω be an arbitrary domain in $\mathbb{R}^n$ and let Q^0_{lr}, Q^1_{lr} be the Whitney cubes according to (2.83) (in immaterial modification of (2.22)). Let $\varrho = \{\varrho_{lr}\}$ be a related resolution of unity with (2.54), (2.55). We are interested in wavelet expansions for the spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ according to Definition 2.14 with (2.56)–(2.58). We rely on the L-version of the wavelet expansion for $F^s_{pq}(\mathbb{R}^n)$ as described at the beginning of Section 1.3.2 applied to

$$f = \sum_{l=0}^{\infty} \sum_r \varrho_{lr} f, \quad \varrho_{lr} f \in F^s_{pq}(\mathbb{R}^n). \tag{2.95}$$

We fix $L \in \mathbb{N}$ so that we have (2.86), (2.87) for the wavelets in (2.85). The crucial point is the wavelet expansion of $\varrho_{lr} f$ according to [T06], Proposition 4.17, p. 206, based on the homogeneity property (2.43) for the spaces $\bar{F}^s_{pq}(U_\lambda)$ in the first line in (2.40). But afterwards one can follow the arguments in [T06] verbatim. One obtains intrinsic wavelet expansions

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m(f)\, 2^{-jn/2}\, \Psi^j_{G,m} \tag{2.96}$$

for $f \in F^{s,\mathrm{rloc}}_{pq}(\Omega)$. We do not repeat the details. But we fix the outcome. Recall that the functions $\Psi^j_{G,m}$ are real. One has

$$\lambda^{j,G}_m(f) = 2^{jn/2} \int_{\mathbb{R}^n} f(x)\, \Psi^j_{G,m}(x)\, \mathrm{d}x, \quad (j,G,m) \in S^{\Omega,1}, \tag{2.97}$$

and

$$\lambda^{j,G}_m(f) = 2^{jn/2} \int_{\mathbb{R}^n} \varrho^j_m(x)\, f(x)\, \Psi^j_{G,m}(x)\, \mathrm{d}x, \quad (j,G,m) \in S^{\Omega,2}, \tag{2.98}$$

where ϱ^j_m are some real C^∞ functions with

$$\operatorname{supp} \varrho^j_m \subset Q_{jr}, \quad |D^\gamma \varrho^j_m(x)| \le c_\gamma\, 2^{j|\gamma|}, \quad \gamma \in \mathbb{N}^n_0, \tag{2.99}$$

and f is extended outside of Ω by zero. Here Q_{jr} has the same meaning as in (2.83) with $r = r(m)$ as in (2.89). Let χ_{jm} be the characteristic function of Q_{jr} with $r = r(m)$ in (2.88), (2.89). We need the Ω-version of f^s_{pq} in Definition 1.18. Let $f^{s,\Omega}_{pq}$ be the collection of all sequences

$$\lambda = \{\lambda^{j,G}_m \in \mathbb{C} : (j,G,m) \in S^\Omega\} \tag{2.100}$$

such that

$$\|\lambda \,|\, f^{s,\Omega}_{pq}\| = \Big\| \Big(\sum_{(j,G,m)\in S^\Omega} 2^{jsq}\, \big|\lambda^{j,G}_m\, \chi_{jm}(\cdot)\big|^q \Big)^{1/q} \,|\, L_p(\Omega) \Big\| \tag{2.101}$$

is finite. Recall that $F^s_{\infty\infty} = B^s_{\infty\infty}$.

Theorem 2.23. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $0 < p \le \infty$, $0 < q \le \infty$ (with $q = \infty$ if $p = \infty$) and $\sigma_{pq} < s < u \in \mathbb{N}$. Let*

$$\{\Psi^j_{G,m} : (j,G,m) \in S^\Omega\} = \Psi^{1,\Omega} \cup \Psi^{2,\Omega} \tag{2.102}$$

be the above real intrinsic wavelet system (2.91), (2.92) based on (1.87), (1.88) and (2.85)–(2.87). Let v be as in (2.70), (2.71). Then $f \in L_v(\Omega)$ is an element of $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ if,and only if, it can be represented by

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^j_{G,m}, \quad \lambda \in f^{s,\Omega}_{pq}. \tag{2.103}$$

Furthermore,

$$\| f \,| F^{s,\mathrm{rloc}}_{pq}(\Omega)\| \sim \inf \|\lambda \,| f^{s,\Omega}_{pq}\| \tag{2.104}$$

where the infimum is taken over all representations (2.103). Any $f \in F^{s,\mathrm{rloc}}_{pq}(\Omega)$ can be represented by (2.96) with (2.97), (2.98) and

$$\|\lambda(f) \,| f^{s,\Omega}_{pq}\| \sim \| f \,| F^{s,\mathrm{rloc}}_{pq}(\Omega)\| \tag{2.105}$$

(*equivalent quasi-norms*).

Proof. This is an extension of [T06], Theorem 4.22 (i), p. 212, from $\widetilde{F}^s_{pq}(\Omega) = F^{s,\mathrm{rloc}}_{pq}(\Omega)$, where Ω is a bounded Lipschitz domain and where $p, q, u = k$ are restricted by [T06], (4.112), p. 212, to the above situation. Now we rely on the improved wavelet expansion Theorem 1.20 in $\mathbb{R}^n$, which results in $s < u$ and its L-version mentioned above. Otherwise one can follow the arguments in [T06]. In particular, (2.103) is an atomic representation for f in $\mathbb{R}^n$ (no moment conditions are required) and after multiplication with ϱ_{lr} also for $\varrho_{lr} f$. One obtains

$$\| f \,| F^{s,\mathrm{rloc}}_{pq}(\Omega)\|^p \sim \sum_{l,r} \|\varrho_{lr} f \,| F^s_{pq}(\mathbb{R}^n)\|^p \le c \, \|\lambda \,| f^{s,\Omega}_{pq}\|^p. \tag{2.106}$$

The crucial point is the representability of $f \in F^{s,\mathrm{rloc}}_{pq}(\Omega)$ by (2.96), (2.105) with (2.104) as a consequence. But this is the same as in [T06] (reduction to $\mathbb{R}^n$, and clipped together via refined localisation). □

Remark 2.24. We add a comment about the convergence of (2.103). First we remark that (2.103) converges unconditionally in $S'(\mathbb{R}^n)$. If $p < \infty$ and $v < \infty$ in (2.70), (2.71) then (2.103) converges absolutely (and hence unconditionally) in $L_v(\Omega)$. If $p < \infty, q < \infty$, then (2.103) converges unconditionally in $F^{s,\mathrm{rloc}}_{pq}(\Omega)$. If $p < \infty, q = \infty$, then one has at least unconditional convergence in $F^{\sigma,\mathrm{rloc}}_{pp}(\Omega)$ with $\sigma_{pq} < \sigma < s$. Recall that $F^s_{\infty\infty} = \mathcal{C}^s$. If Ω is bounded and $p = q = \infty$ then (2.103) converges unconditionally in $\mathcal{C}^{\sigma,\mathrm{rloc}}(\Omega)$ with $0 < \sigma < s$. If Ω is unbounded and $p = q = \infty$ then one has unconditional convergence in $\mathcal{C}^{\sigma,\mathrm{rloc}}(\Omega \cap K_R)$ where K_R is a ball centred at the origin and of radius R ($\to \infty$).

Remark 2.25. By (2.88)–(2.90) and (2.86) we have for any $(j,G,m) \in S^\Omega$ that

$$\operatorname{supp} \Psi^j_{G,m} \subset Q_{lr} \subset \Omega \tag{2.107}$$

for some $l \le j$ and some r. Then (2.96)–(2.98) is an intrinsic stable *frame representation*, where stable refers to (2.103), (2.104). What is meant by a *frame* will also be discussed later on in Remark 5.11 in greater detail. As mentioned in connection with (2.93), (2.94) the main part $\Psi^{1,\Omega}$ with $(j,G,m) \in S^{\Omega,1}$ consists of pairwise orthogonal Daubechies wavelets. It is an unconditional basis ($p<\infty$, $q<\infty$) on the *complemented* subspace $F^{s,\mathrm{rloc},1}_{pq}(\Omega)$ spanned by $\Psi^{1,\Omega}$,

$$F^{s,\mathrm{rloc}}_{pq}(\Omega) = F^{s,\mathrm{rloc},1}_{pq}(\Omega) \oplus F^{s,\mathrm{rloc},2}_{pq}(\Omega) \tag{2.108}$$

where the corresponding projection operators can be constructed by (2.96) (with $S^{\Omega,1}$ in place of S^Ω) and (2.105). As for the residual part we have the orthogonality (2.93), (2.94) and

$$F^{s,\mathrm{rloc},2}_{pq}(\Omega) \subset C^u(\Omega). \tag{2.109}$$

Furthermore, $F^{s,\mathrm{rloc},2}_{pq}(\Omega)$ is locally finite-dimensional (which means that for any bounded domain ω with $\overline{\omega} \subset \Omega$, there are only finitely many functions $\Psi^j_{G,m}$ with $(j,G,m) \in S^{\Omega,2}$ such that $\operatorname{supp} \Psi^j_{G,m} \cap \overline{\omega} \neq \emptyset$). This may justify calling (2.102) a *para-basis* (with the usual notational caution if $p<\infty$, $q=\infty$ or even $p=q=\infty$).

Remark 2.26. The proof of Theorem 1.20 is based on the observation that the Daubechies wavelets may serve simultaneously as atoms and as kernels of local means satisfying the needed moment conditions. This is also the case for the above wavelets $\Psi^j_{G,m}$ if $(j,G,m) \in S^{\Omega,1}$, hence $G \in \{F,M\}^{n*}$ (main wavelet system (2.91)) because we have always (1.88). But this does not apply to the scaled down scaling functions $\Psi^j_{\overline{F},m}$ with $G = \overline{F} = (F,\dots,F)$. As a consequence one has now only a para-basis. However it will be one of the main aims of what follows to convert (2.102) into an orthonormal basis which is a u-wavelet system according to Definition 2.4. Then the basic wavelets (2.32) and the interior wavelets (2.33) coincide with the above main wavelet system $\Psi^{1,\Omega}$ in (2.91), complemented by some $\Psi^0_{G,m}$, whereas the boundary wavelets (2.34), (2.35) are obtained by local orthonormalisation from the remaining elements of the residual wavelet system $\Psi^{2,\Omega}$ in (2.92).

Remark 2.27. One has always the continuous embedding

$$F^{s,\mathrm{rloc}}_{pq}(\Omega) \hookrightarrow \widetilde{F}^s_{pq}(\Omega) \tag{2.110}$$

for p, q, s as in Theorem 2.23, where $\widetilde{F}^s_{pq}(\Omega)$ are the spaces introduced in Definition 2.1. This follows from (2.103), (2.104) and the atomic arguments in the proof of Theorem 2.23. It will be one of the main points of what follows to show that

$$F^{s,\mathrm{rloc}}_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega), \quad 0<p,q\le\infty,\ s>\sigma_{pq}, \tag{2.111}$$

($q = \infty$ if $p = \infty$) for huge classes of domains, the so-called E-thick domains, including bounded C^∞ domains and bounded Lipschitz domains. We mentioned a special case of (2.111) in (2.82). This observation will pave the way to develop a theory of wavelet bases for large classes of the spaces $\tilde{A}^s_{pq}(\Omega)$ and $A^s_{pq}(\Omega)$ according to Definition 2.1 in E-thick domains. This is the subject of Chapter 3.

2.3.3 Wavelet para-bases in $L_p(\Omega)$, $1 < p < \infty$

One may ask whether Theorem 2.23 can be extended to other refined localisation spaces according to Definition 2.14 having the homogeneity property (2.43) (which are two of the main ingredients of the above proof with a reference to [T06]). But this causes some trouble. To obtain (2.106) we used that (2.103) is an atomic decomposition in $\mathbb{R}^n$ (after appropriate normalisation). Since $s > \sigma_{pq}$ no moment conditions for the atoms are needed. But if $s \le 0$ then Theorem 1.7 requires some moment conditions (1.31). The elements $\Psi^j_{G,m} \in \Psi^{1,\Omega}$ of the main wavelet system (2.91) fit in this scheme, but not the elements $\Psi^j_{\tilde{F},m} \in \Psi^{2,\Omega}$ with $j \in \mathbb{N}$ of the residual wavelet system (2.92) briefly mentioned in Remark 2.26. But there is a remarkable exception.

Based on the well-known Paley–Littlewood property

$$F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n), \quad 1 < p < \infty, \tag{2.112}$$

one has by Definitions 2.1, 2.14 that

$$F^0_{p,2}(\Omega) = \widetilde{F}^0_{p,2}(\Omega) = F^{0,\mathrm{rloc}}_{p,2}(\Omega) = L_p(\Omega), \quad 1 < p < \infty, \tag{2.113}$$

for arbitrary domains Ω with $\Omega \neq \mathbb{R}^n$. We need $f^{0,\Omega}_{p,2}$ according to (2.100), (2.101) with $s = 0$, $q = 2$.

Theorem 2.28. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $1 < p < \infty$ and $u \in \mathbb{N}$. Let*

$$\left\{\Psi^j_{G,m} : (j,G,m) \in S^\Omega\right\} = \Psi^{1,\Omega} \cup \Psi^{2,\Omega} \tag{2.114}$$

be the same intrinsic wavelet system as in (2.102) based on (1.87), (1.88) and (2.85)–(2.87). Then $L_p(\Omega)$ is the collection of all locally integrable functions f (either with respect to $\mathbb{R}^n$ or to Ω) which can be represented by

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^j_{G,m}, \quad \lambda \in f^{0,\Omega}_{p,2}. \tag{2.115}$$

Furthermore,

$$\| f \,| L_p(\Omega)\| \sim \inf \|\lambda \,| f^{0,\Omega}_{p,2}\|, \tag{2.116}$$

where the infimum is taken over all representations (2.115). Any $f \in L_p(\Omega)$ can be represented by (2.96) with (2.97), (2.98) and

$$\|\lambda(f) \,| f^{0,\Omega}_{p,2}\| \sim \| f \,| L_p(\Omega)\| \tag{2.117}$$

(equivalent norms).

Proof. Obviously, $L_p(\Omega)$ has the refined localisation property, hence (2.106) with L_p in place of F^s_{pq} on the left-hand side, and the homogeneity (2.43) with L_p instead of $\overline{A}^0_{pq}$. Then one obtains (2.96) with (2.97), (2.98) and (2.117). Let f be given by (2.115). In contrast to the atomic representation (2.103) in $F^s_{pq}(\mathbb{R}^n)$ with $s > \sigma_{pq}$ where no moment conditions for the corresponding atoms are required the situation is now different. According to Theorem 1.7 one needs for atomic representations

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\, a_{jm} \tag{2.118}$$

in $L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n)$ with $1 < p < \infty$ first moment conditions

$$\int_{\mathbb{R}^n} a_{jm}(x)\, \mathrm{d}x = 0, \quad j \in \mathbb{N},\ m \in \mathbb{Z}^n. \tag{2.119}$$

We split (2.115) according to (2.88)–(2.92) into the main part and the residual part

$$\begin{aligned} f &= \sum_{(j,G,m)\in S^{\Omega,1}} 2^{-jn/p}\, \lambda^{j,G}_m\, a^G_{jm} + \sum_{(j,G,m)\in S^{\Omega,2}} 2^{-jn/p}\, \lambda^{j,G}_m\, a^G_{jm} \\ &= f_1 + f_2 \end{aligned} \tag{2.120}$$

with

$$a^G_{jm} = 2^{jn(\frac{1}{p}-\frac{1}{2})}\, \Psi^j_{G,m}. \tag{2.121}$$

If $\Psi^j_{G,m} \in \Psi^{1,\Omega}$ then a^G_{jm} are normalised atoms in $L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n)$ satisfying (2.119) with a^G_{jm} in place of a_{jm}. We split (2.100) naturally in λ^1 and λ^2,

$$\lambda^l = \{\lambda^{j,G}_m \in \mathbb{C} : (j,G,m) \in S^{\Omega,l}\}, \quad l = 1,2. \tag{2.122}$$

Then it follows by Theorem 1.7 that

$$\| f_1 \,|L_p(\Omega)\| = \| f_1 \,|L_p(\mathbb{R}^n)\| \le c\, \big\|\lambda^1 \,| f^{0,\Omega}_{p,2}\big\|. \tag{2.123}$$

Furthermore, for the residual part f_2 one has

$$\| f_2 \,|L_p(\Omega)\|^p \le c \sum_{(j,G,m)\in S^{\Omega,2}} 2^{-jn}\, |\lambda^{j,G}_m|^p \sim \big\|\lambda^2 \,| f^{0,\Omega}_{pp}\big\|^p \sim \big\|\lambda^2 \,| f^{0,\Omega}_{p,2}\big\|^p, \tag{2.124}$$

where we used (2.101) and the structure of $S^{\Omega,2}$ discussed in Remark 2.25. This proves

$$\| f \,|L_p(\Omega)\| \le c\, \big\|\lambda \,| f^{0,\Omega}_{p,2}\big\|. \tag{2.125}$$

Together with (2.117) one has (2.116). □

Remark 2.29. In other words, one obtains for $L_p(\Omega)$, $1 < p < \infty$, in arbitrary domains Ω in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ intrinsic wavelet para-bases of any smoothness $u \in \mathbb{N}$ in terms of the Littlewood–Paley assertion (2.117). In particular one has common wavelet para-bases for all spaces covered by Theorems 2.23 and 2.28 as long as $s < u$. Later one we convert these wavelet para-bases into orthonormal bases which are u-wavelet systems according to Definition 2.4. We refer to Section 2.4.2.

Remark 2.30. One may ask what happens if (2.114) is not based on the smooth functions ψ_F, ψ_M according to (1.87), (1.88), but on related Haar functions as described in [T06], Proposition 1.54, p. 28. Then we obtain suitably adapted systems of type H_n as in [T06], Theorem 1.58, p. 29. By this theorem and the above considerations one has intrinsic Haar bases (not only para-bases) for $L_p(\Omega)$. This observation can be extended to those spaces $F^{s,\mathrm{rloc}}_{pp}(\Omega) = B^{s,\mathrm{rloc}}_{pp}(\Omega)$ which are covered both by Theorem 2.23 and [T06], Theorem 1.58, p. 29 , in particular

$$\frac{n}{n+1} < p < \infty, \quad \sigma_p < s < \min\Big(1, \frac{1}{p}\Big). \tag{2.126}$$

We return to this point later on and refer to Section 2.5.1.

2.4 Wavelet bases

2.4.1 Orthonormal wavelet bases in $L_2(\Omega)$

Again let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Recall that domain means open set. So far we have by Theorems 2.23 and 2.28 wavelet para-bases

$$\big\{\Psi^j_{G,m} : (j,G,m) \in S^\Omega\big\} = \Psi^{1,\Omega} \cup \Psi^{2,\Omega} \tag{2.127}$$

in some refined localisation spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ and in $L_p(\Omega)$ with $1 < p < \infty$. Here $\Psi^{1,\Omega}$ and $\Psi^{2,\Omega}$ are the main wavelet system and the residual wavelet system according to (2.88)–(2.92), based on (2.85)–(2.87). These two systems are orthogonal to each other and they span $L_2(\Omega)$,

$$L_2(\Omega) = L_2^{(1)}(\Omega) \oplus L_2^{(2)}(\Omega), \quad L_2^{(l)}(\Omega) = \operatorname{span} \Psi^{l,\Omega}, \tag{2.128}$$

(2.93), (2.94), where $l = 1, 2$. Furthermore, $\Psi^{1,\Omega}$ is an orthonormal basis of $L_2^{(1)}(\Omega)$. A few words about the elements of the residual set $\Psi^{2,\Omega}$ have been said after (2.94) and in Remark 2.25. There is the temptation to apply Schmidt's very classical orthonormalisation as it may be found in textbooks about Hilbert spaces, for example [Yos80], Chapter III, Section 5, p. 88, or [Tri92], pp. 86–87, to $\Psi^{2,\Omega}$. This will be of some help for us locally. But globally it is of little use beyond $L_2(\Omega)$ since it destroys the localisation (support property) of its elements

$$\Psi^j_{G,m}(x) = 2^{(j+L)n/2} \prod_{a=1}^{n} \psi_{G_a}\big(2^{j+L}x_a - m_a\big), \quad (j,G,m) \in S^{\Omega,2}. \tag{2.129}$$

To circumvent these difficulties we use some (one-dimensional) multiresolution arguments, the product structure of (2.129) and a further refinement of the Whitney decomposition (2.83). Let now

$$Q^0_{lr} \subset Q^1_{lr} \subset Q^2_{lr} \subset Q^3_{lr} \subset Q_{lr}, \quad l \in \mathbb{N}_0, \ r = 1, 2, \ldots, \tag{2.130}$$

be concentric open cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates centred at $2^{-l} m^r$ for some $m^r \in \mathbb{Z}^n$ with the respective side-lengths $2^{-l}, 5 \cdot 2^{-l-2}, 6 \cdot 2^{-l-2}, 7 \cdot 2^{-l-2}, 2^{-l+1}$. In other words, we insert now the cubes Q^3_{lr} in the construction at the beginning of Section 2.3.1. Otherwise, Q^0_{lr} are the same pairwise disjoint cubes as there with (2.84). Recall that $|l - l'| \le 1$ for two cubes Q^1_{lr}, $Q^1_{l'r'}$ having a non-empty intersection. We fix now $L \in \mathbb{N}_0$ in (2.129) such that we have (2.86) with Q^3_{lr} in place of Q_{lr}, hence

$$\operatorname{supp} \Psi^j_{G,m} \subset Q^3_{lr} \quad \text{if} \quad 2^{-j-L} m \in Q^2_{lr} \quad \text{for } l \in \mathbb{N}_0 \text{ and } j \ge l, \tag{2.131}$$

whereas (2.87) remains unchanged. This refinement ensures that the supports of the wavelets related to cubes Q^0_{lr} and $Q^0_{l+2,r'}$ are disjoint. Instead of this explicit construction one could simply say that L is chosen sufficiently large (but fixed once and for all).

Definition 2.31. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and let $u \in \mathbb{N}$. Then

$$\left\{ \Phi^j_r : \ j \in \mathbb{N}_0; \ r = 1, \ldots, N_j \right\} \quad \text{with } N_j \in \overline{\mathbb{N}} \tag{2.132}$$

is called an orthonormal u-wavelet basis in $L_2(\Omega)$ if it is both an u-wavelet system according to Definition 2.4 and an orthonormal basis in $L_2(\Omega)$.

Remark 2.32. In Definition 2.4 we marked not only the required smoothness $u \in \mathbb{N}$ in the wavelets (2.27) based on (2.28), (2.29) but also L. Now we adopt the position that L is chosen sufficiently large as described and fixed. This may justify our omission of L in what follows. Similarly as in (2.91), (2.92) we put

$$\Phi^{1,\Omega} = \left\{ \text{interior wavelets } \Phi^j_r \right\} \tag{2.133}$$

and

$$\Phi^{2,\Omega} = \left\{ \text{basic and boundary wavelets } \Phi^j_r \right\}. \tag{2.134}$$

Theorem 2.33. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. For any $u \in \mathbb{N}$ there are orthonormal u-wavelet bases in $L_2(\Omega)$ according to Definition 2.31.*

Proof. Step 1. Let

$$\left\{ \Psi^j_{G,m} : \ (j, G, m) \in S^\Omega \right\} = \Psi^{1,\Omega} \cup \Psi^{2,\Omega} \tag{2.135}$$

be the same wavelet system as in Theorem 2.28 based on (1.87), (1.88) and (2.85)–(2.87) now modified by (2.131). We have (2.128). Then

$$\Phi^{1,\Omega} = \Psi^{1,\Omega} \quad \text{is an orthonormal basis in } L^{(1)}_2(\Omega). \tag{2.136}$$

Furthermore, $L_2^{(2)}(\Omega)$ is orthogonal to $L_2^{(1)}(\Omega)$. Hence we must find an orthonormal system $\Phi^{2,\Omega}$ according to (2.134) such that it spans $L_2^{(2)}(\Omega)$. The basic wavelets Φ_r^0 in (2.32) are the elements $\Psi^0_{G,m}$ in (2.92), based on (2.89) (recall that $S_0^{\Omega,1} = \emptyset$). They fit in the above scheme and need not to be considered. Hence it is sufficient to care for the elements with $j \in \mathbb{N}$ in (2.92). These are boundary elements. In other words one must find an orthonormalisation of the elements of $\Psi^{2,\Omega}$ with $j \in \mathbb{N}$ resulting in boundary wavelets according to Definition 2.4 (iii).

Step 2. First we deal with the one-dimensional model case $\Omega = (-\infty, 0)$. Let Q^0_{lr} in (2.130) be an interval centred at $2^{-l}r$ for some suitable negative integer r and of side-length 2^{-l}. The related residual wavelets according to (2.129) are given by

$$\psi^l_m(x) = 2^{(l+L)/2}\, \psi_G(2^{l+L}x - m), \quad m \in \mathbb{Z}, \tag{2.137}$$

with $\psi_G = \psi_F$ (scaling function) or $\psi_G = \psi_M$ (wavelet). Of interest is the case where the right endpoint of Q^0_{lr}, hence $x_{l,r} = 2^{-l}r + 2^{-l-1}$, is the left endpoint of an admitted Whitney interval $Q^0_{l+1,r'}$. Then some functions

$$\psi_G\big(2^{l+L}x - m\big) \quad \text{and} \quad \psi_F\big(2^{l+1+L}x - m'\big) \tag{2.138}$$

centred near $x_{l,r}$ may not be orthogonal. (Recall that we always assume that L is large ensuring that everything is local near $x_{l,r}$). The decisive argument comes from the multiresolution property

$$\psi_G\big(2^{l+L}x - m\big) = \sum_{t\in\mathbb{Z}} c^G_{l,m,t}\, \psi_F\big(2^{l+1+L}x - t\big), \tag{2.139}$$

where only terms with

$$\operatorname{supp} \psi_F\big(2^{l+1+L}\cdot - t\big) \subset \operatorname{supp} \psi_G\big(2^{l+L}\cdot - m\big) \tag{2.140}$$

are of interest. Otherwise one has $c^G_{l,m,t} = 0$. Hence (2.139) is a finite sum where only terms with

$$\big|2^{-l-1-L}t - 2^{-l-L}m\big| \le c\, 2^{-L-l}, \quad \text{hence } |t - 2m| \le 2c, \tag{2.141}$$

for some $c > 0$ are needed. Furthermore, replacing x by $2x$ in (2.139) it follows that $c^G_{l,m,t} = c^G_{m,t}$ is independent of l. The coefficients $c^G_{l,m}$ are also translation invariant, $c^G_{m,t} = c^G_{m+m_0,t+2m_0}$. Hence all coefficients in (2.139) coincide with finitely many real numbers. Let $\mathbb{Z}_m$ be the collection of all $t \in \mathbb{Z}$ with (2.141). Let $G = F$ in (2.137), (2.139) and let temporarily $\psi = \psi_F$. Then one obtains by (2.137), (2.139) that

$$\psi^l_m = \sum_{t\in\mathbb{Z}'_m} d_{m,t}\, \psi^{l+1}_t + \sum_{t\in\mathbb{Z}''_m} d_{m,t}\, \psi^{l+1}_t = \psi'^{\,l+1}_m + \psi''^{\,l+1}_m, \tag{2.142}$$

where $\mathbb{Z}_m = \mathbb{Z}'_m \cup \mathbb{Z}''_m$ is decomposed such that $\psi'^{\,l+1}_m$ collects all terms on the right-hand side of (2.139) with (2.140) and $\psi^{l+1}_t \in \Psi^{2,\Omega}$. The remaining terms $\psi''^{\,l+1}_m$ are not only orthogonal to ψ'^{l+1}_m by the orthogonality of the translated scaling functions, but also to all other terms of $\Psi^{1,\Omega}$ and $\Psi^{2,\Omega}$ not involved in (2.139). Of course, one can replace $G = F$ in (2.137), (2.139) in the above arguments by $G = M$. The resulting functions ψ''^{l+1}_m are orthogonal to all other functions in $\Psi^\Omega = \Psi^{1,\Omega} \cup \Psi^{2,\Omega}$, but they may be not orthogonal among themselves. But now one can apply the orthogonalisation procedure mentioned above between the formulas (2.128) and (2.129). In this way one eliminates possible linear dependences (stepping from frames to bases). Then one obtains functions which are orthogonal among themselves and to all other functions from Ψ^Ω not involved in this local procedure. This can be done at any point $2^{-l}r + 2^{-l-1}$. Together with the unchanged elements of $\Psi^{2,\Omega}$ one obtains an orthonormal system in

$$L_2^{(2)}(\Omega) \quad \text{with } \Omega = (-\infty, 0) \tag{2.143}$$

which can be identified with (2.134) collecting basic and boundary wavelets according to Definition 2.4 (i), (iii). The above discussion about the coefficients $c^G_{l,m,t}$ in (2.139) justifies (2.35). Together with (2.136) one obtains an orthonormal basis in $L_2(\Omega)$ with $\Omega = (-\infty, 0)$ which is an u-wavelet basis according to Definition 2.31. Since everything is local this proves also the theorem for arbitrary domains Ω in $\mathbb{R}$.

Step 3. The corresponding assertion for arbitrary domains Ω in $\mathbb{R}^n$ with $n \geq 2$ can be reduced to the 1-dimensional case. Let $\Psi^j_{G,m}$ with $j \in \mathbb{N}$ be an element of the residual system $\Psi^{2,\Omega}$ according to (2.129) based on the Whitney cubes in (2.130) (excluding again the basic wavelets). We assume that the right face with respect to the x_1-direction, say, $x_1 = 2^{-l}r + 2^{-l-1}$ as in the above 1-dimensional model case, is part of the left face of an admitted Whitney cube $Q^0_{l+1,r'}$. Then one can apply the 1-dimensional orthogonalisation procedure with respect to the x_1-factors according to Step 2. By the product structure of $\Psi^j_{G,m}$ in (2.129) it follows from Fubini's theorem that this is also an n-dimensional orthogonalisation. Applying (2.142) to the x_1-direction one obtains after multiplication with the remaining $n-1$ directions terms of type

$$\psi^{l+1}_t(x_1)\, \Psi^l_{G,m'}(x'), \qquad x' = (x_2, \ldots, x_n). \tag{2.144}$$

Using again the multiresolution property one obtains the $(n-1)$-dimensional counterpart of (2.139),

$$\Psi^l_{G,m'}(x') = \sum_{t' \in \mathbb{Z}^{n-1}} d^{G,n}_{l,m',t'}\, \Psi^{l+1}_{\bar{F},t'}(x'), \tag{2.145}$$

with a counterpart of (2.140). Here $\Psi^{l+1}_{\bar{F},t}(x')$ with $\bar{F} = (F, \ldots, F)$ are the same dilated scaling functions as in Definition 2.4. Inserted in (2.144) one obtains a linear combination of terms

$$\psi^{l+1}_t(x_1)\, \Psi^{l+1}_{\bar{F},t'}(x'). \tag{2.146}$$

However it is not guaranteed that each element in (2.146) is an element in the wavelet expansion with respect to $Q^0_{l+1,r'}$. Decomposing (2.145) as in (2.142) there may remain linear combinations of terms

$$\psi^{l+1}_t(x_1)\, \Psi^{l+1}_{\bar{F},t'}(x') \tag{2.147}$$

which are orthogonal to all other functions of $\Psi^{2,\Omega}$, involved in the above procedure or not, and also to $\Psi^{1,\Omega}$. However this undesirable effect can only happen near x_v-faces of $Q^0_{l+1,r'}$ with $v = 2, \dots, n$, hence near edges (with respect to x_1 and x_v). In order to clarify what is going on we assume first $n = 2$. Of course in the above arguments x_1 and x_2 can change their roles making clear that the indicated effect can happen only in corner points. Now one can proceed with the orthogonalisation as described above and orthogonalise the remaining functions locally around isolated corners. Then they are orthogonal among themselves and to all other functions in $\Psi^{2,\Omega}$ and $\Psi^{1,\Omega}$. They have the desired structure (2.34), (2.35) as boundary wavelets. For $n \geq 3$ one can apply this argument iteratively as follows. If the wavelets considered for orthogonalisation are not near a corner-point of the related cube then there is at least one harmless direction of coordinates (playing the role of x_2 at the beginning of this step). Orthogonalisation for the remaining $(n-1)$ directions may be assumed by induction. Finally one can orthogonalise the remaining functions locally around isolated corners as described above in case of $n = 2$. The discussion about the coefficients in (2.139) has an n-dimensional counterpart being independent of dilation and translations. This ensures the existence of the numbers K and D in (2.34), (2.35). □

Remark 2.34. Both in case of the para-bases (2.102) based on (2.88)–(2.92) on the one hand and in case of u-wavelet bases according to Definitions 2.4, 2.31 on the other hand one tries to preserve as much as possible from the $\mathbb{R}^n$-wavelets as described in Section 1.2.1 when switching from $\mathbb{R}^n$ to domains Ω in $\mathbb{R}^n$. It is quite clear that one has to pay a price. The $\mathbb{R}^n$-wavelets depend heavily on translations and dyadic dilatations. There are no counterparts for domains (with exception of wavelets on the n-torus $\mathbb{T}^n$ as considered in Section 1.3). There is a huge literature about wavelets on domains. We refer to [Mal99], [CDV00], [CDV04], [Dah97], [Dah01], [Coh03], [HoL05]. But as far as we can see the constructions suggested there have little in common with our approach. In case of para-bases one has now the residual wavelet system $\Psi^{2,\Omega}$ in (2.92) and in particular the scaled down scaling functions $\Psi^j_{\overline{F},m}$ (in the notation used in Definition 2.4) which spoil the orthogonality. First steps in this direction had been done in [T06], Section 4.2 (based on [Tri07a]). This has been extended in [Tri07b]. Using the rather particular properties of wavelets (multiresolution analysis, product structure) one can restore the lost orthogonality at the expense of the lattice structure. These are the boundary wavelets in Definition 2.4 (iii). Then one obtains bases which preserve at least in a qualitative way typical ingredients of wavelets such as localisation, dyadic refinements and also cancellations as far as basic and interior wavelets are concerned. This is presented here for the first time. It is one of the main aims of this exposition to develop this theory and its far-reaching consequences.

Remark 2.35. We fix the outcome such that it can be used as a starting point of what follows beyond L_2. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and let $u \in \mathbb{N}$. For suitably chosen numbers $L \in \mathbb{N}$, $K \in \mathbb{N}$, $D > 0$, positive constants $c_1, \dots, c_4$ and approximate lattices $\mathbb{Z}_\Omega$ there are orthonormal real u-wavelet bases in $L_2(\Omega)$,

$$\left\{\Phi^j_r : j \in \mathbb{N}_0;\ r = 1, \dots, N_j\right\} \quad \text{where } N_j \in \overline{\mathbb{N}}, \tag{2.148}$$

according to Definitions 2.4, 2.31 and Theorem 2.33. Then $f \in L_2(\Omega)$ can be expanded as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \big(f, \Phi_r^j\big)\, \Phi_r^j = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda_r^j\, 2^{-jn/2}\, \Phi_r^j \tag{2.149}$$

with

$$\lambda_r^j = \lambda_r^j(f) = 2^{jn/2} \int_{\mathbb{R}^n} f(x)\, \Phi_r^j(x)\, \mathrm{d}x, \tag{2.150}$$

adapted notationally to our later needs with the real L_∞-normalised functions $2^{-jn/2}\Phi_r^j$ (where L can be neglected). To justify the integration over $\mathbb{R}^n$ in (2.150) one may assume that f is extended outside of Ω by zero. Of course,

$$\sum_{j=0}^{\infty} \sum_{r=1}^{N_j} 2^{-jn}\, |\lambda_r^j|^2 = \|f\,|L_2(\Omega)\|^2. \tag{2.151}$$

Recall that both the basic wavelets Φ_r^0 in (2.32) with $G \in \{F, M\}^{n*}$ and the interior wavelets Φ_r^j in (2.33) have the cancellation property

$$\int_{\mathbb{R}^n} x^\beta\, \Phi_r^j(x)\, \mathrm{d}x = 0 \quad \text{if } |\beta| < u. \tag{2.152}$$

This follows from (1.88) and (2.27).

2.4.2 Wavelet bases in $L_p(\Omega)$ and $F^{s,\mathrm{rloc}}_{pq}(\Omega)$

We obtained the orthonormal u-wavelet basis in Theorem 2.33 converting the residual wavelet system $\Psi^{2,\Omega}$ in (2.92) into the basic and the boundary wavelets (2.32), (2.34), (2.35) by strictly local arguments. This makes clear that one can now transfer Theorems 2.23 and 2.28 from wavelet para-bases to u-wavelet bases. Let $f^s_{pq}(\mathbb{Z}_\Omega)$ be the sequence spaces as introduced in Definition 2.6. As usual $L_1^{\mathrm{loc}}(\Omega)$ collects all complex-valued Lebesgue-measurable functions in Ω such that

$$\int_\omega |f(x)|\, \mathrm{d}x < \infty \quad \text{for any bounded domain } \omega \text{ with } \overline{\omega} \subset \Omega. \tag{2.153}$$

Theorem 2.36. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $1 < p < \infty$ and $u \in \mathbb{N}$. Let*

$$\big\{\Phi_r^j : j \in \mathbb{N}_0;\ r = 1, \ldots, N_j\big\} \quad \text{with } N_j \in \overline{\mathbb{N}}, \tag{2.154}$$

be an orthonormal u-wavelet basis in $L_2(\Omega)$ according to Theorem 2.33 and Definition 2.31. Then $L_p(\Omega)$ is the collection of all $f \in L_1^{\mathrm{loc}}(\Omega)$ which can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda_r^j\, 2^{-jn/2}\, \Phi_r^j, \quad \lambda \in f^0_{p,2}(\mathbb{Z}_\Omega). \tag{2.155}$$

Furthermore, $\{\Phi_r^j\}$ is an unconditional basis in $L_p(\Omega)$. If $f \in L_p(\Omega)$ then the representation (2.155) is unique with $\lambda = \lambda(f)$,

$$\lambda_r^j(f) = 2^{jn/2}\left(f, \Phi_r^j\right) = 2^{jn/2} \int_\Omega f(x)\,\Phi_r^j(x)\,dx \tag{2.156}$$

and

$$I : f \mapsto \lambda(f) = \left\{2^{jn/2}\left(f, \Phi_r^j\right)\right\} \tag{2.157}$$

*is an isomorphic map of $L_p(\Omega)$ **onto** $f_{p,2}^0(\mathbb{Z}_\Omega)$,*

$$\|f\,|L_p(\Omega)\| \sim \|\lambda(f)\,|f_{p,2}^0(\mathbb{Z}_\Omega)\| \tag{2.158}$$

(*equivalent norms*).

Proof. We split f given by (2.155) as in (2.120) into $f = f_1 + f_2$, where f_1 collects the terms with the interior wavelets (2.33), and f_2 the terms with the basic and the boundary wavelets (2.32), (2.34), (2.35). We have (2.152) with $\beta = 0$ for the terms in f_1. Then we are in the same position as in the proof of Theorem 2.28. Hence, (2.155) converges unconditionally in $L_p(\mathbb{R}^n)$ and hence in $L_p(\Omega)$. Furthermore by construction of the orthonormal u-wavelet basis in $L_2(\Omega)$, an element f can be represented by (2.115) if, and only if, it can be represented by (2.155). If $f \in D(\Omega)$ then this representation is unique with (2.150), and hence (2.156) (recall that the Φ_r^j's are real and compactly supported in Ω). Since $D(\Omega)$ is dense in $L_p(\Omega)$ one obtains by completion the unique representation (2.155), (2.156), and also (2.158). By the above comments it follows that I in (2.157) is the indicated isomorphic map. □

Remark 2.37. One may consider (2.158) as a Paley–Littlewood characterisation of $L_p(\Omega)$ in terms of wavelet bases. As mentioned in Remark 2.30 there is a counterpart of the above theorem in terms of Ω-adapted Haar bases. We refer to Section 2.5.1

Recall that $f_{pq}^s(\mathbb{Z}_\Omega)$ are the sequence spaces according to Definition 2.6.

Theorem 2.38. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $F_{pq}^{s,\mathrm{rloc}}(\Omega)$ be the spaces as introduced in Definition 2.14 with (2.56)–(2.58),*

$$0 < p \le \infty, \quad 0 < q \le \infty, \; s > \sigma_{pq}, \tag{2.159}$$

($q = \infty$ if $p = \infty$). Let $\{\Phi_r^j\}$ in (2.154) be an orthonormal u-wavelet basis in $L_2(\Omega)$ according to Theorem 2.33 and Definition 2.31 with $s < u \in \mathbb{N}$. Let v be as in (2.70), (2.71). Then $f \in L_v(\Omega)$ is an element of $F_{pq}^{s,\mathrm{rloc}}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda_r^j\, 2^{-jn/2}\, \Phi_r^j, \quad \lambda \in f_{pq}^s(\mathbb{Z}_\Omega). \tag{2.160}$$

*Furthermore if $f \in F_{pq}^{s,\mathrm{rloc}}(\Omega)$ then the representation (2.160) is unique with $\lambda = \lambda(f)$ as in (2.156) and I in (2.157) is an isomorphic map of $F_{pq}^{s,\mathrm{rloc}}(\Omega)$ **onto** $f_{pq}^s(\mathbb{Z}_\Omega)$,*

$$\|f\,|F_{pq}^{s,\mathrm{rloc}}(\Omega)\| \sim \|\lambda(f)\,|f_{pq}^s(\mathbb{Z}_\Omega)\| \tag{2.161}$$

(*equivalent quasi-norms*). *If* $p < \infty$, $q < \infty$ *then* $\{\Phi_r^j\}$ *is an unconditional basis in* $F_{pq}^{s,\mathrm{rloc}}(\Omega)$.

Proof. By the same arguments as in the proof of Theorem 2.23 it follows that (2.160) is an atomic decomposition in $\mathbb{R}^n$ (no moment conditions are needed). One has as a consequence $f \in F_{pq}^{s,\mathrm{rloc}}(\Omega)$. Furthermore since $f \in F_{pq}^{s,\mathrm{rloc}}(\Omega)$ can be represented by (2.103) it can also be represented by (2.160). By (2.70), (2.71) it is also a representation in $L_v(\Omega)$. Then it follows from Theorem 2.36 that this representation is unique with (2.156). Finally one obtains the remaining assertions from Theorem 2.23 and the above comments. □

Remark 2.39. To what extent the spaces $F_{pq}^{s,\mathrm{rloc}}(\Omega)$ with (2.159) in arbitrary domains Ω in $\mathbb{R}^n$ are of self-contained interest is not so clear. On the one hand we have satisfactory characterisations of these spaces in Theorem 2.18 and Corollary 2.20. On the other hand we described in Remark 2.17 the little history of these spaces with (2.63) as the main assertion. The motivation to deal with the spaces $F_{pq}^{s,\mathrm{rloc}}(\Omega)$ comes from the question under which circumstances the more natural spaces $\widetilde{F}_{pq}^{s}(\Omega)$ have the refined localisation property, expressed by (2.63). This cannot be true for general domains, but it will be valid in so-called E-thick domains. This is one of the main points of the following Chapter 3. We refer in particular to Proposition 3.10.

2.5 Complements

2.5.1 Haar bases

We complement the above considerations by a few topics which are a little bit outside the main stream of this exposition. We will be brief. First we add some comments about Haar bases. The classical forerunners of the Daubechies wavelets (1.87), (1.88) on $\mathbb{R}$ are the *Haar wavelet* h_M,

$$h_M(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1/2, \\ -1 & \text{if } 1/2 < x \le 1, \\ 0 & \text{if } x \notin [0,1], \end{cases} \tag{2.162}$$

and the scaling function h_F, $h_F(x) = |h_M(x)|$ (the characteristic function of the interval $[0,1]$). This very first example of a multiresolution analysis resulting in an orthonormal basis in $L_2(\mathbb{R})$ goes back to Haar, [Haar10]. Otherwise the n-dimensional machinery described in Section 1.2.1 can be applied. In particular,

$$\{H_{G,m}^j : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\} \tag{2.163}$$

with

$$H_{G,m}^j(x) = 2^{jn/2} \prod_{r=1}^{n} h_{G_r}(2^j x_r - m_r), \quad G \in G^j,\ m \in \mathbb{Z}^n, \tag{2.164}$$

and $j \in \mathbb{N}_0$, *is an orthonormal basis in* $L_2(\mathbb{R}^n)$. The corresponding expansion is given by

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_m^{j,G} \, 2^{-jn/2} \, H_{G,m}^j \tag{2.165}$$

with

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2} \, \big(f, H_{G,m}^j\big). \tag{2.166}$$

This is the counterpart of (1.93), (1.94). Let b_{pq}^s and f_{pq}^s be the same sequence spaces as in Definition 1.18. Then Theorem 1.20 can be complemented as follows. Recall that $\sigma_p = n\big(\frac{1}{p} - 1\big)_+$.

Theorem 2.40. *Let $H_{G,m}^j$ be the Haar wavelets (2.164).*

(i) *Let $1 < p < \infty$. Then $f \in S'(\mathbb{R}^n)$ is an element of $L_p(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \lambda_m^{j,G} \, 2^{-jn/2} \, H_{G,m}^j, \quad \lambda \in f_{p,2}^0, \tag{2.167}$$

unconditional convergence being in $L_p(\mathbb{R}^n)$. The representation (2.167) is unique with $\lambda_m^{j,G} = \lambda_m^{j,G}(f)$ as in (2.166). Furthermore,

$$I : f \mapsto \big\{2^{jn/2} \, \big(f, H_{G,m}^j\big)\big\} \tag{2.168}$$

is an isomorphic map of $L_p(\mathbb{R}^n)$ onto $f_{p,2}^0$ and $\big\{H_{G,m}^j\big\}$ is an unconditional basis in $L_p(\mathbb{R}^n)$.

(ii) *Let $0 < p < \infty$ and*

$$\sigma_p < s < \min\Big(1, \frac{1}{p}\Big). \tag{2.169}$$

Then $f \in S'(\mathbb{R}^n)$ is an element of $B_{pp}^s(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j,G,m} \lambda_m^{j,G} \, 2^{-jn/2} \, H_{G,m}^j, \quad \lambda \in b_{pp}^s, \tag{2.170}$$

unconditional convergence being in $B_{pp}^s(\mathbb{R}^n)$. The representation (2.170) is unique with $\lambda_m^{j,G} = \lambda_m^{j,G}(f)$ as in (2.166). Furthermore I in (2.168) is an isomorphic map of $B_{pp}^s(\mathbb{R}^n)$ onto b_{pp}^s and $\big\{H_{G,m}^j\big\}$ is an unconditional basis in $B_{pp}^s(\mathbb{R}^n)$.

Proof. Step 1. Part (i) is a famous result of real analysis. A proof may be found in [Woj97], Section 8.3. It goes back to J. Marcinkiewicz, [Mar37] (one-dimensional case), who in turn relied on [Pal32]. The above formulation is a little bit pedantic. But we wanted to keep the assertion in the context of Theorem 1.20.

Step 2. The proof of Theorem 1.20 relies on atomic representations as described in Theorem 1.7 and the identification of $\lambda_m^{j,G}(f)$ in (2.166) with local means according to Theorem 1.15. We modify (1.111) by

$$k_{jm}^G = 2^{jn/2}\, H_{G,m}^j, \quad j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n. \tag{2.171}$$

Then we can apply Theorem 1.15 with $A = 0$ and $B = 1$, what requires $\sigma_p < s < 1$. The atomic Theorem 1.7 cannot be applied immediately. If $s < 1/p$ then characteristic functions of cubes are elements of $B_{pp}^s(\mathbb{R}^n) = F_{pp}^s(\mathbb{R}^n)$. If

$$0 < p < \infty, \quad \sigma_p < s < \sigma < \frac{1}{p}, \tag{2.172}$$

then one can apply the homogeneity assertion (2.43) based on the first lines in (2.40), (2.41) or (2.47), covered by [CLT07]. One obtains that

$$\|H_{G,m}^j \,|B_{pp}^\sigma(\mathbb{R}^n)\| \sim 2^{j(\sigma-s)}\, \|H_{G,m}^j \,|B_{pp}^s(\mathbb{R}^n)\|. \tag{2.173}$$

Now it follows from [T06], Theorem 2.13, that (2.170) is an expansion by non-smooth atoms (after correct normalisation). Hence one can also apply Step 2 of the proof of Theorem 1.20 as long as $\sigma_p < s < 1/p$. Condition (2.169) covers the needed restrictions for s. As in Step 1 of the proof of Theorem 1.20 we refer for technicalities to [T06], Section 3.1.3, Theorem 3.5. □

Remark 2.41. We refer to [T06], Theorem 1.58, Remark 1.59, pp. 29–30, where we discussed in detail under which circumstances $\{H_{G,m}^j\}$ is a basis in $B_{pq}^s(\mathbb{R}^n)$. In particular the restriction (2.169) is natural. We gave here a new independent proof of part (ii) of the above theorem for two reasons. First the assertions in [T06], Theorem 1.58, do not cover that the basis $\{H_{G,m}^j\}$ is unconditional and that I in (2.168) is an isomorphic map onto b_{pp}^s. Secondly the above arguments are of the same type as in the proof of Theorem 1.20. One may ask whether part (ii) of the above theorem can be extended to other spaces $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$. This is surely possible, but not done so far. The sharp assertions about local means according to Theorem 1.15 fit in such a scheme. Also the needed homogeneity (2.43) is available. But one has to extend the theory of non-smooth atoms as treated in [T06], Theorem 2.13, from $B_{pp}^s(\mathbb{R}^n)$ to $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$. The generalisation of the above theorem from $B_{pp}^s(\mathbb{R}^n)$ to $B_{pq}^s(\mathbb{R}^n)$ can also be done by real interpolation.

Our main concern in Chapter 2 is not with spaces on $\mathbb{R}^n$, but with spaces on arbitrary domains Ω in $\mathbb{R}^n$. Taking Theorem 2.40 as the Haar version of Theorem 1.20 we are now in the same position as in case of the Daubechies wavelets in Ω. Some assertions are even simpler now. Let Q_{lr}^0 be the same pairwise disjoint open Whitney cubes as in (2.22), (2.23) and let χ_{lr} be their characteristic functions. First we complement Theorem 2.16. To be consistent with our previous notation we prefer now F_{pp}^s instead of B_{pp}^s. Recall that $F_{pp}^s = B_{pp}^s$.

Proposition 2.42. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $0 < p < \infty$,*

$$\sigma_p < s < \min\left(1, \frac{1}{p}\right), \tag{2.174}$$

and

$$\max(1, p) < v < \infty, \quad s - \frac{n}{p} > -\frac{n}{v}. \tag{2.175}$$

Let $F^{s,\mathrm{rloc}}_{pp}(\Omega)$ be the same space as in Definition 2.14. Then

$$F^{s,\mathrm{rloc}}_{pp}(\Omega) = \left\{ f \in L_v(\Omega) : \| f \,|F^{s,\mathrm{rloc}}_{pp}(\Omega)\|_\chi < \infty \right\} \tag{2.176}$$

with

$$\| f \,|F^{s,\mathrm{rloc}}_{pp}(\Omega)\|_\chi = \left(\sum_{l=0}^{\infty} \sum_r \|\chi_{lr} f \,|F^s_{pp}(\mathbb{R}^n)\|^p \right)^{1/p}, \tag{2.177}$$

(*equivalent quasi-norms*).

Proof. We have (2.70), (2.71) which justifies to replace $D'(\Omega)$ in (2.56) by $L_v(\Omega)$ in order to avoid technical problems caused by the multiplication with characteristic functions. Otherwise we are in the same position as in the proof of Theorem 2.16. We need the pointwise multiplier assertion according to Theorem 2.13 for the characteristic functions χ_{lr}. As there one can reduce this question by homogeneity to the problem of whether the characteristic function of the unit cube is a pointwise multiplier in $B^s_{pp}(\mathbb{R}^n) = F^s_{pp}(\mathbb{R}^n)$ with p, s as above. But this is a well-known assertion and may be found in [T83], Theorem 2.8.7, p. 158. □

Now we can complement the wavelet para-bases and wavelet bases originating from Daubechies wavelets by Haar bases adapted to Ω (being again an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$). Let Q^0_{lr} be the open Whitney cubes according to (2.22), (2.23) of side-length 2^{-l} now with the left corner $2^{-l}m^r$ for some $m^r \in \mathbb{Z}^n$ (in immaterial modification of the previous setting). Let h_M as in (2.162) and $h_F(x) = |h_M(x)|$ as before. We extend (2.164) to $G \in \{F, M\}^n$, hence

$$H^j_{G,m}(x) = 2^{jn/2} \prod_{a=1}^{n} h_{G_a}(2^j x_a - m_a), \quad G \in \{F, M\}^n, \ m \in \mathbb{Z}^n, \tag{2.178}$$

and $j \in \mathbb{N}_0$, which is the counterpart of (2.85). We extend Q^0_{lr} with (2.22), (2.23) by its left and lower faces,

$$Q^1_{lr} = 2^{-l}m^r + 2^{-l}\{x \in \mathbb{R}^n : 0 \le x_a < 1\}, \quad \overline{Q^1_{lr}} = \overline{Q^0_{lr}}. \tag{2.179}$$

Then the adapted modifications of (2.88)–(2.92) are given by

$$S^{\Omega,1}_j = \{F, M\}^{n*} \times \{m \in \mathbb{Z}^n : 2^{-j}m \in Q^1_{lr} \text{ for some } l < j, \text{some } r\}, \tag{2.180}$$

$$S_j^{\Omega,2} = \{F, M\}^n \times \{m \in \mathbb{Z}^n : 2^{-j} m \in Q^1_{jr} \text{ for some } r\}, \tag{2.181}$$

$$S^\Omega = S^{\Omega,1} \cup S^{\Omega,2}, \quad S^{\Omega,1} = \bigcup_{j=1}^{\infty} S_j^{\Omega,1}, \quad S^{\Omega,2} = \bigcup_{j=0}^{\infty} S_j^{\Omega,2}, \tag{2.182}$$

and

$$H^\Omega = \left\{ H^j_{G,m} : (j, G, m) \in S^\Omega \right\}. \tag{2.183}$$

The Haar system inside an admitted cube Q^0_{lr} is the scaled down by the factor 2^{-l} and translated Haar system for the unit cube. By construction H^Ω is the orthonormal union of these Haar systems in all admitted cubes Q^0_{jr}. One obtains in particular the following complement of Theorem 2.33.

Proposition 2.43. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Then H^Ω according to* (2.183) *is an orthonormal basis in $L_2(\Omega)$.*

Proof. This follows from the corresponding assertion for the unit cube in $\mathbb{R}^n$ and the above constructions and comments. □

Now we are in the same position as in Section 2.3. But instead of the para-basis $\{\Psi^j_{G,m}\}$ in (2.102) we have now the orthonormal basis H^Ω in (2.183). Additional local orthogonalisations as in Section 2.4 are not necessary now. As in (2.100), (2.101) let

$$f^{s,\Omega}_{pq} = \left\{ \lambda : \|\lambda \,|\, f^{s,\Omega}_{pq}\| < \infty \right\} \tag{2.184}$$

with

$$\|\lambda \,|\, f^{s,\Omega}_{pq}\| = \Big\| \Big(\sum_{(j,G,m)\in S^\Omega} 2^{jsq} \, |\lambda^{j,G}_m \, \chi_{jm}(\cdot)|^q \Big)^{1/q} |L_p(\Omega) \Big\|, \tag{2.185}$$

where

$$\lambda = \left\{ \lambda^{j,G}_m \in \mathbb{C} : (j, G, m) \in S^\Omega \right\}. \tag{2.186}$$

If $0 < p = q < \infty$ then

$$\|\lambda \,|\, f^{s,\Omega}_{pp}\| \sim \Big(\sum_{(j,G,m)\in S^\Omega} 2^{j(s-\frac{n}{p})p} \, |\lambda^{j,G}_m|^p \Big)^{1/p}. \tag{2.187}$$

Theorem 2.44. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let H^Ω be the orthonormal Haar basis in $L_2(\Omega)$ according to* (2.183).

(i) *Let $1 < p < \infty$. Then $L_p(\Omega)$ is the collection of all $f \in L_1^{\mathrm{loc}}(\Omega)$ which can be represented as*

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m \, 2^{-jn/2} \, H^j_{G,m}, \quad \lambda \in f^{0,\Omega}_{p,2}, \tag{2.188}$$

unconditional convergence being in $L_p(\Omega)$. *Furthermore,* H^Ω *is an unconditional basis in* $L_p(\Omega)$. *If* $f \in L_p(\Omega)$ *then the representation* (2.188) *is unique with* $\lambda = \lambda(f)$,

$$\lambda_m^{j,G}(f) = 2^{jn/2}\,\big(f, H^j_{G,m}\big) = 2^{jn/2} \int_\Omega f(x)\, H^j_{G,m}(x)\,\mathrm{d}x \tag{2.189}$$

and

$$I: f \mapsto \lambda(f) = \big\{2^{jn/2}\big(f, H^j_{G,m}\big)\big\} \tag{2.190}$$

is an isomorphic map of $L_p(\Omega)$ *onto* $f^{0,\Omega}_{p,2}$,

$$\|f\,|L_p(\Omega)\| \sim \|\lambda(f)\,|f^{0,\Omega}_{p,2}\| \tag{2.191}$$

(*equivalent norms*).

(ii) *Let* $0 < p < \infty$,

$$\sigma_p < s < \min\Big(1, \frac{1}{p}\Big) \tag{2.192}$$

and v *as in* (2.175). *Let* $F^{s,\mathrm{rloc}}_{pp}(\Omega)$ *be the same space as in Definition* 2.14. *Then* $F^{s,\mathrm{rloc}}_{pp}(\Omega)$ *is the collection of all* $f \in L_v(\Omega)$ *which can be represented by*

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m\, 2^{-jn/2}\, H^j_{G,m}, \quad \lambda \in f^{s,\Omega}_{pp}, \tag{2.193}$$

according to (2.184)–(2.187), *unconditional convergence being in* $F^{s,\mathrm{rloc}}_{pp}(\Omega)$. *Furthermore,* H^Ω *is an unconditional basis in* $F^{s,\mathrm{rloc}}_{pp}(\Omega)$. *If* $f \in F^{s,\mathrm{rloc}}_{pp}(\Omega)$ *then the representation* (2.193) *is unique with* $\lambda = \lambda(f)$ *as in* (2.189) *and* I *in* (2.190) *is an isomorphic map of* $F^{s,\mathrm{rloc}}_{pp}(\Omega)$ *onto* $f^{s,\Omega}_{pp}$,

$$\big\|f\,|F^{s,\mathrm{rloc}}_{pp}(\Omega)\big\| \sim \big\|\lambda(f)\,|f^{s,\Omega}_{pp}\big\| \tag{2.194}$$

(*equivalent quasi-norms*).

Proof. Based on Theorem 2.40 and Propositions 2.42, 2.43 one can follow the arguments from Sections 2.3, 2.4. □

2.5.2 Wavelet bases in Lorentz and Zygmund spaces

We mainly deal in this book with the spaces $A^s_{pq}(\Omega)$ on domains, their wavelet representations and related extension problems. But we outline a few related topics. So far we have wavelet bases for the Lebesgue spaces L_p with $1 < p < \infty$ in $\mathbb{R}^n$, covered by Theorems 1.20, 2.40 (i) with $L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n)$, and in arbitrary domains Ω with $\Omega \neq \mathbb{R}^n$ according to Theorems 2.36, 2.44 (i). We wish to extend these wavelet representations to the

$$\text{Lorentz spaces}\ \ L_{p,q}(\Omega), \quad 1 < p < \infty,\ 1 \le q < \infty, \tag{2.195}$$

and the

$$\text{Zygmund spaces } L_p(\log L)_a(\Omega), \quad 1 < p < \infty, \ a \in \mathbb{R}. \tag{2.196}$$

We restrict ourselves to a description and refer for details, proofs, and further references to [Tri08].

Let Ω be either $\mathbb{R}^n$ or an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$, let $|\Gamma|$ be the Lebesgue measure of a (Lebesgue measurable) set Γ in $\mathbb{R}^n$ and let f be a complex-valued Lebesgue measurable, a.e. finite function in Ω. Then

$$\mu_f(\varrho) = |\{x \in \Omega : |f(x)| > \varrho\}|, \quad \varrho \geq 0, \tag{2.197}$$

is the distribution function of f and f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf\{\varrho : \mu_f(\varrho) \leq t\}, \quad t \geq 0, \tag{2.198}$$

its (decreasing) rearrangement. Details and properties may be found in [BeS88], [EdE04] and [Har07]. The Lorentz spaces $L_{p,q}(\Omega)$ and the Zygmund spaces $L_p(\log L)_a(\Omega)$ can be defined for all $0 < p, q \leq \infty$, $a \in \mathbb{R}$. But we restrict ourselves to those spaces which are of interest for us.

Definition 2.45. Let $1 < p < \infty$.

(i) Let Ω be either $\mathbb{R}^n$ or an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $1 \leq q < \infty$. Then

$$L_{p,q}(\Omega) = \big\{ f \in L_1^{\mathrm{loc}}(\Omega) : \|f \,|L_{p,q}(\Omega)\| < \infty \big\} \tag{2.199}$$

with

$$\|f \,|L_{p,q}(\Omega)\| = \left(\int_0^{|\Omega|} \big(t^{1/p} f^*(t)\big)^q \, \frac{\mathrm{d}t}{t} \right)^{1/q}. \tag{2.200}$$

(ii) Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $|\Omega| < \infty$. Let $a \in \mathbb{R}$. Then

$$L_p(\log L)_a(\Omega) = \big\{ f \in L_1(\Omega) : \|f \,|L_p(\log L)_a(\Omega)\| < \infty \big\} \tag{2.201}$$

with

$$\|f \,|L_p(\log L)_a(\Omega)\| = \left(\int_0^{|\Omega|} (1 + |\log t|)^{ap} \ f^*(t)^p \, \mathrm{d}t \right)^{1/p}. \tag{2.202}$$

Remark 2.46. The above Lorentz spaces $L_{p,q}(\Omega)$ and Zygmund spaces $L_p(\log L)_a(\Omega)$ are Banach spaces (although (2.200) and (2.202) are only respective equivalent quasi-norms). Recall that $L_{p,q}(\Omega)$ can be obtained by *real interpolation*

$$L_{p,q}(\Omega) = \big(L_{p_0}(\Omega), L_{p_1}(\Omega)\big)_{\theta,q}, \quad \tfrac{1}{p} = \tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1}, \tag{2.203}$$

where $1 < p_0 < p < p_1 < \infty$, [T78], Section 1.18.6. The above Zygmund spaces $L_p(\log L)_a(\Omega)$ can also be reduced by *extrapolation* to Lebesgue spaces $L_{\tilde{p}}(\Omega)$ with $1 < \tilde{p} < \infty$. Here one needs $|\Omega| < \infty$. This may be found in [ET96], Section 2.6.2, with a reference to [EdT95]. It is just this reduction of the above Lorentz and Zygmund spaces to Lebesgue spaces $L_p(\Omega)$ with $1 < p < \infty$ which paves the way to transfer wavelet representations for the Lebesgue spaces to Lorentz and Zygmund spaces. As said we restrict ourselves here to a description referring for details to [Tri08]. We concentrate ourselves first to u-wavelet bases in domains $\Omega \neq \mathbb{R}^n$ and incorporate afterwards $\Omega = \mathbb{R}^n$ and Haar bases. We need the straightforward extension of Definition 2.6 using the same notation as there. In particular, χ_{jr} is the characteristic function of the ball $B(x^j_r, c_2 2^{-j})$ in (2.31).

Definition 2.47. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ ($|\Omega| < \infty$ for the Zygmund spaces) and let $\mathbb{Z}_\Omega$ be as in (2.24)–(2.26). Let $1 < p < \infty$, $1 \le q < \infty$, $a \in \mathbb{R}$. Then $\Lambda_{pq}(\mathbb{Z}_\Omega)$ is the collection of all sequences

$$\lambda = \{\lambda^j_r \in \mathbb{C} : j \in \mathbb{N}_0;\ r = 1, \dots, N_j\}, \quad N_j \in \bar{\mathbb{N}}, \tag{2.204}$$

such that

$$\|\lambda \,|\Lambda_{p,q}(\mathbb{Z}_\Omega)\| = \Big\| \Big(\sum_{j=0}^{\infty} \sum_{r=1}^{N_j} |\lambda^j_r \, \chi_{jr}(\cdot)|^2 \Big)^{1/2} |L_{p,q}(\Omega)\Big\| < \infty \tag{2.205}$$

and $\Lambda_p(\log \Lambda)_a(\mathbb{Z}_\Omega)$ is the collection of all sequences (2.204) such that

$$\|\lambda \,|\Lambda_p(\log \Lambda)_a(\mathbb{Z}_\Omega)\| = \Big\| \Big(\sum_{j=0}^{\infty} \sum_{r=1}^{N_j} |\lambda^j_r \, \chi_{jr}(\cdot)|^2 \Big)^{1/2} |L_p(\log L)_a(\Omega)\Big\| < \infty. \tag{2.206}$$

Remark 2.48. Recall that

$$L_{p,p}(\Omega) = L_p(\log L)_0(\Omega) = L_p(\Omega), \quad 1 < p < \infty. \tag{2.207}$$

Similarly one has

$$\Lambda_{p,p}(\mathbb{Z}_\Omega) = \Lambda_p(\log \Lambda)_0(\mathbb{Z}_\Omega) = f^0_{p,2}(\mathbb{Z}_\Omega) \tag{2.208}$$

according to the above definition and (2.38).

Theorem 2.49. *Let $1 < p < \infty$ and $u \in \mathbb{N}$.*

(i) *Let $1 \le q < \infty$ and let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let*

$$\{\Phi^j_r : j \in \mathbb{N}_0;\ r = 1, \dots, N_j\} \quad \text{with } N_j \in \bar{\mathbb{N}}, \tag{2.209}$$

be an orthonormal u-wavelet basis in $L_2(\Omega)$ according to Theorem 2.33 and Definition 2.31. Then $L_{p,q}(\Omega)$ is the collection of all $f \in L^{\mathrm{loc}}_1(\Omega)$ which can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r \, 2^{-jn/2} \, \Phi^j_r, \quad \lambda \in \Lambda_{p,q}(\mathbb{Z}_\Omega). \tag{2.210}$$

Furthermore, $\{\Phi_r^j\}$ *is an unconditional basis in* $L_{p,q}(\Omega)$*. If* $f \in L_{p,q}(\Omega)$ *then the representation* (2.210) *is unique with* $\lambda = \lambda(f)$*,*

$$\lambda_r^j(f) = 2^{jn/2}\left(f, \Phi_r^j\right) = 2^{jn/2} \int_\Omega f(x)\, \Phi_r^j(x)\, \mathrm{d}x \tag{2.211}$$

and

$$I : f \mapsto \lambda(f) = \left\{2^{jn/2}\left(f, \Phi_r^j\right)\right\} \tag{2.212}$$

is an isomorphic map of $L_{p,q}(\Omega)$ *onto* $\Lambda_{pq}(\mathbb{Z}_\Omega)$*,*

$$\|f \,|L_{p,q}(\Omega)\| \sim \|\lambda(f)\,|\Lambda_{p,q}(\mathbb{Z}_\Omega)\| \tag{2.213}$$

(*equivalent quasi-norms*).

(ii) *Let* $a \in \mathbb{R}$ *and let* Ω *be an arbitrary domain in* $\mathbb{R}^n$ *with* $|\Omega| < \infty$*. Then* $L_p(\log L)_a(\Omega)$ *is the collection of all* $f \in L_1(\Omega)$ *which can be represented as*

$$f = \sum_{j=0}^{\infty}\sum_{r=1}^{N_j} \lambda_r^j\, 2^{-jn/2}\, \Phi_r^j, \quad \lambda \in \Lambda_p(\log\Lambda)_a(\mathbb{Z}_\Omega). \tag{2.214}$$

Furthermore, $\{\Phi_r^j\}$ *is an unconditional basis in* $L_p(\log L)_a(\Omega)$*. If* $f \in L_p(\log L)_a(\Omega)$ *then the representation* (2.214) *is unique with* (2.211) *and* I *in* (2.212) *is an isomorphic map of* $L_p(\log L)_a(\Omega)$ *onto* $\Lambda_p(\log\Lambda)_a(\mathbb{Z}_\Omega)$ *with a counterpart of* (2.213).

Remark 2.50. This is the extension of Theorem 2.36 from Lebesgue spaces to Lorentz and Zygmund spaces. Basically one proves the above theorem for the Lorentz spaces by the interpolation (2.203) of corresponding Lebesgue spaces (and its sequence counterpart). Similarly for the Zygmund spaces using extrapolation. But this requires some care and is shifted to [Tri08].

Although it is quite clear that there is a counterpart of the above theorem in terms of Haar bases, generalising Theorems 2.40 (i), 2.44 (i) it seems to be reasonable to give an explicit formulation. Let H^Ω be the orthonormal Haar basis according to Proposition 2.43 and (2.183). We incorporate now $\mathbb{R}^n$ identifying H^Ω in this case with $H^{\mathbb{R}^n} = \{H^j_{G,m}\}$ according to (2.163). We need the Lorentz and Zygmund counterparts of the sequence spaces $f^{s,\Omega}_{pq}$ in (2.184), (2.186) originating from (2.101), (2.100) with $s = 0$ and $q = 2$. Let Ω be an arbitrary domain in $\mathbb{R}^n$ (where $\Omega = \mathbb{R}^n$ is now admitted) and let $1 < p < \infty$, $1 \le q < \infty$. Then Λ^Ω_{pq} is the collection of all sequences

$$\lambda = \left\{\lambda^{j,G}_m \in \mathbb{C} : (j, G, m) \in S^\Omega\right\} \tag{2.215}$$

such that

$$\|\lambda\,|\Lambda^\Omega_{pq}\| = \Big\| \Big(\sum_{(j,G,m)\in S^\Omega} |\lambda^{j,G}_m\, \chi_{jm}(\cdot)|^2 \Big)^{1/2} |L_{p,q}(\Omega) \Big\| < \infty. \tag{2.216}$$

Similarly $\Lambda_p(\log \Lambda)^\Omega_a$ with $1 < p < \infty$, $a \in \mathbb{R}$, $|\Omega| < \infty$, is the collection of all sequences (2.215) such that

$$\|\lambda \,|\Lambda_p(\log \Lambda)^\Omega_a\| = \Big\| \Big(\sum_{(j,G,m)\in S^\Omega} |\lambda^{j,G}_m \, \chi_{jm}(\cdot)|^2\Big)^{1/2} |L_p(\log L)_a(\Omega)\Big\| < \infty. \tag{2.217}$$

This is the adapted counterpart of Definition 2.47. Now Theorems 2.44 (i) and 2.49 can be complemented as follows.

Corollary 2.51. *Let $1 < p < \infty$.*

(i) *Let $1 \le q < \infty$ and let Ω be an arbitrary domain $\mathbb{R}^n$ (where $\Omega = \mathbb{R}^n$ is admitted). Let*

$$H^\Omega = \big\{H^j_{G,m} : \ (j,G,m) \in S^\Omega\big\} \tag{2.218}$$

be the orthonormal Haar basis in $L_2(\Omega)$ according to Proposition 2.43 and (2.183) interpreted in the indicated way if $\Omega = \mathbb{R}^n$. Then $L_{p,q}(\Omega)$ is the collection of all $f \in L^{\mathrm{loc}}_1(\Omega)$ which can be represented as

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m \, 2^{-jn/2} \, H^j_{G,m}, \quad \lambda \in \Lambda^\Omega_{pq}. \tag{2.219}$$

Furthermore, H^Ω is an unconditional basis in $L_{p,q}(\Omega)$. If $f \in L_{p,q}(\Omega)$ then the representation (2.219) is unique with $\lambda = \lambda(f)$ as in (2.189) and I in (2.190) is an isomorphic map of $L_{p,q}(\Omega)$ onto Λ^Ω_{pq}.

(ii) *Let $a \in \mathbb{R}$ and let Ω be an arbitrary domain in $\mathbb{R}^n$ with $|\Omega| < \infty$. Then $L_p(\log L)_a(\Omega)$ is the collection of all $f \in L_1(\Omega)$ which can be represented by*

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m \, 2^{-jn/2} \, H^j_{G,m}, \quad \lambda \in \Lambda_p(\log \Lambda)^\Omega_a. \tag{2.220}$$

Furthermore, H^Ω is an unconditional basis in $L_p(\log L)_a(\Omega)$. If f is contained in $L_p(\log L)_a(\Omega)$ then the representation (2.220) is unique with $\lambda = \lambda(f)$ as in (2.189) and I in (2.190) is an isomorphic map of $L_p(\log L)_a(\Omega)$ onto $\Lambda_p(\log \Lambda)^\Omega_a$.

Remark 2.52. The proof of Theorem 2.49 in [Tri08] extends Theorem 2.36 by linear interpolation, some extrapolation and duality from Lebesgue spaces to Lorentz and Zygmund spaces. Similarly one obtains Corollary 2.51 from Theorem 2.44 (i). Orthogonal wavelet bases in more general rearrangement invariant spaces on $\mathbb{R}^n$ have been considered in [Soa97] using sub-linear real interpolation. One may ask whether our arguments can be based on more general interpolation assertions in order to extend Theorem 2.49 and Corollary 2.51 to other Orlicz spaces and rearrangement invariant spaces. Zygmund spaces may be called logarithmic L_p-spaces. There are corresponding logarithmic Sobolev spaces and several extensions based on logarithmic interpolation spaces, closely connected with extrapolation arguments. This theory started in [EdT95], [ET96], Section 6.2, and has been developed afterwards in [EdT98], [CFT04], [CFMM07]. It remains to be seen whether one can use these techniques also in connection with what follows here to construct wavelet bases in logarithmic versions of function spaces of type A^s_{pq}.

2.5.3 Constrained wavelet expansions for Sobolev spaces

Later on we deal in Section 4.3.2 with constrained wavelet expansions in spaces $A^s_{pq}(\Omega)$ with $0 < p, q \le \infty$ ($p < \infty$ for F-spaces) and $s > 0$ in Lipschitz domains Ω in $\mathbb{R}^n$. There one finds also further discussions about this topic. We describe here a somewhat curious counterpart for Sobolev spaces in arbitrary domains.

Definition 2.53. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $1 < p < \infty$ and $k \in \mathbb{N}$. Then

$$W^k_p(\Omega) = \big\{ f \in D'(\Omega) : \ \| f \,|W^k_p(\Omega)\| < \infty \big\}, \tag{2.221}$$

$$\| f \,|W^k_p(\Omega)\| = \sum_{|\alpha| \le k} \| D^\alpha f \,| L_p(\Omega)\|. \tag{2.222}$$

Remark 2.54. Recall that

$$F^k_{p,2}(\mathbb{R}^n) = W^k_p(\mathbb{R}^n), \quad 1 < p < \infty, \ k \in \mathbb{N}. \tag{2.223}$$

Since $F^k_{p,2}(\Omega)$, denoted temporarily as $\mathcal{W}^k_p(\Omega)$, is defined according to (2.4), (2.5) by restriction one has the continuous embedding

$$\mathcal{W}^k_p(\Omega) \hookrightarrow W^k_p(\Omega), \quad 1 < p < \infty, \ k \in \mathbb{N}. \tag{2.224}$$

The question whether these two spaces coincide might be called the *non-linear extension problem*. But more interesting is the (linear) *extension problem*, which means the search for linear bounded operators

$$\mathrm{ext}: \ \mathcal{W}^k_p(\Omega) \hookrightarrow W^k_p(\mathbb{R}^n) \quad \text{with } \mathrm{ext}\, f|\Omega = f \tag{2.225}$$

or

$$\mathrm{Ext}: \ W^k_p(\Omega) \hookrightarrow W^k_p(\mathbb{R}^n) \quad \text{with } \mathrm{Ext}\, f|\Omega = f, \tag{2.226}$$

where $g|\Omega$ has the same meaning as in (2.2). Later on we deal in detail with (linear) extensions of type (2.225) for all spaces $A^s_{pq}(\Omega)$. Some of the references given there apply also to (2.226). For the spaces $\mathcal{W}^k_p(\Omega)$ one has the Sobolev embedding

$$\mathcal{W}^k_p(\Omega) \hookrightarrow L_q(\Omega), \quad k - \tfrac{n}{p} \ge -\tfrac{n}{q}, \ 1 < p \le q < \infty, \tag{2.227}$$

obtained by restriction of a corresponding assertion on $\mathbb{R}^n$. It is well known that the two spaces in (2.224) coincide for large classes of domains. This applies in particular to bounded Lipschitz domains, where one has also (2.225), (2.226). For arbitrary domains Ω in $\mathbb{R}^n$ one cannot expect that the two spaces in (2.224) coincide and also the counterpart

$$W_p^k(\Omega) \hookrightarrow L_q(\Omega), \quad k - \frac{n}{p} \geq -\frac{n}{q}, \; 1 < p \leq q < \infty, \tag{2.228}$$

of (2.227) need not hold for all q, but maybe for some q (near p). Curiously enough one can address questions of type (2.228) in terms of wavelet expansions.

So far we have for the Sobolev spaces $W_p^{k,\mathrm{rloc}}(\Omega)$ according to (2.74) both the equivalent norms in Corollary 2.20 and the orthonormal u-wavelet bases with $u > k$ in Theorem 2.38. Nothing like this can be expected for $W_p^k(\Omega)$ in arbitrary domains. However there is a curious substitute.

The elements Φ_r^j of any orthonormal u-wavelet basis in $L_2(\Omega)$ according to Theorem 2.33 and Definitions 2.31, 2.4 have compact supports in Ω. This applies also to the elements $H_{G,m}^j$ of the Haar basis H^Ω in Proposition 2.43 and (2.183). In particular,

$$D^\alpha \Phi_r^j \quad \text{and} \quad D^\alpha H_{G,m}^j, \quad \alpha \in \mathbb{N}_0^n, \tag{2.229}$$

have also compact supports in Ω. (It does not matter whether one looks at (2.229) as elements of $S'(\mathbb{R}^n)$ or $D'(\Omega)$.) If $f \in W_p^k(\Omega)$ according to Definition 2.53 then one can apply Theorem 2.36 (and similarly Theorem 2.44 (i)) to each $D^\alpha f \in L_p(\Omega)$ with $|\alpha| \leq k$,

$$D^\alpha f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda_r^j (D^\alpha f)\, 2^{-jn/2}\, \Phi_r^j, \tag{2.230}$$

where

$$\lambda_r^j(D^\alpha f) = 2^{jn/2} \big(D^\alpha f, \Phi_r^j\big) = (-1)^{|\alpha|} 2^{jn/2} \big(f, D^\alpha \Phi_r^j\big). \tag{2.231}$$

It may happen that $D^\alpha \Phi_r^j$ are no longer regular distributions, but this does not matter. Let for $k \in \mathbb{N}$,

$$\lambda(f)^k = \big\{\lambda_r^j(f)^k \in \mathbb{R}_+ : \; j \in \mathbb{N}_0; \; r = 1, \ldots, N_j\big\} \tag{2.232}$$

with

$$\lambda_r^j(f)^k = 2^{jn/2} \sum_{|\alpha| \leq k} \big|\big(f, D^\alpha \Phi_r^j\big)\big|. \tag{2.233}$$

We use the same notation as in Theorem 2.36.

Theorem 2.55. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $1 < p < \infty$, $k \in \mathbb{N}$ and $u \in \mathbb{N}$. Let*

$$\big\{\Phi_r^j : \; j \in \mathbb{N}_0; \; r = 1, \ldots, N_j\big\} \quad \textit{with } N_j \in \overline{\mathbb{N}}, \tag{2.234}$$

be an orthonormal u-wavelet basis in $L_2(\Omega)$ *as in Theorem 2.33 and Definition 2.31. Then* $W^k_p(\Omega)$ *according to Definition 2.53 is the collection of all* $f \in L_p(\Omega)$ *which can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r(f)\, 2^{-jn/2}\, \Phi^j_r, \quad \lambda(f)^k \in f^0_{p,2}(\mathbb{Z}_\Omega). \tag{2.235}$$

Furthermore,

$$\|f\,|W^k_p(\Omega)\| \sim \|\lambda(f)^k\,|f^0_{p,2}(\mathbb{Z}_\Omega)\|. \tag{2.236}$$

Proof. This is an immediate consequence of Theorem 2.36 combined with (2.230)–(2.233). □

Remark 2.56. The wavelet expansion (2.155), (2.158) with $\lambda(f) \in f^0_{p,2}(\mathbb{Z}_\Omega)$ is now the subject of the additional *constraints* $\lambda(f)^k \in f^0_{p,2}(\mathbb{Z}_\Omega)$. This may justify to call (2.235) a *constrained wavelet expansion* of $f \in W^k_p(\Omega)$. It gives the possibility to decide whether a function $f \in L_p(\Omega)$ or $f \in L^{\mathrm{loc}}_1(\Omega)$ belongs to $W^k_p(\Omega)$ independently of the quality of Ω.

Remark 2.57. As mentioned above one can replace the orthonormal u-wavelet basis $\{\Phi^j_r\}$ in $L_2(\Omega)$ by the Haar basis H^Ω. In other words, $f \in L_p(\Omega)$ belongs to $W^k_p(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m(f)\, 2^{-jn/2}\, H^j_{G,m}, \quad \lambda(f)_k \in f^{0,\Omega}_{p,2}, \tag{2.237}$$

and

$$\|f\,|W^k_p(\Omega)\| \sim \big\|\lambda(f)_k\,|f^{0,\Omega}_{p,2}\big\| \tag{2.238}$$

in generalisation of Theorem 2.44. Here

$$\lambda(f)_k = \big\{\lambda^{j,G}_m(f)_k \in \mathbb{R}_+ :\ (j,G,m) \in S^\Omega\big\} \tag{2.239}$$

with

$$\lambda^{j,G}_m(f)_k = 2^{jn/2} \sum_{|\alpha|\le k} \big|\big(f, D^\alpha H^j_{G,m}\big)\big|. \tag{2.240}$$

If $|\alpha| \ge 1$ then $D^\alpha H^j_{G,m}$ are δ-distributions and their derivatives on faces and edges of cubes.

Remark 2.58. If Ω is a bounded Lipschitz domain then there is a linear extension operator according to (2.226), $\mathcal{W}^k_p(\Omega) = W^k_p(\Omega)$, and (2.227) with $W^k_p(\Omega)$ in place of $\mathcal{W}^k_p(\Omega)$. One obtains by Theorem 2.55 and (2.227) that

$$\|\lambda(f)\,|f^0_{q,2}(\mathbb{Z}_\Omega)\| \le c\,\|\lambda(f)^k\,|f^0_{p,2}(\mathbb{Z}_\Omega)\| \tag{2.241}$$

for all p, q in (2.227). Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. *Then one has*

$$W_p^k(\Omega) \hookrightarrow L_q(\Omega), \quad k - \tfrac{n}{p} \geq -\tfrac{n}{q},\ 1 < p \leq q < \infty, \tag{2.242}$$

if, and only if, there is a number $c > 0$ *with* (2.241) *for all* $f \in L_p(\Omega)$. (Recall that $f \in L_p(\Omega)$ is an element of $W_p^k(\Omega)$ if the right-hand side of (2.241) is finite.) But it is not clear (to the author) how (2.241) is related to the geometry of the, say, connected domain Ω (beyond bounded Lipschitz domains).

Remark 2.59. According to Definition 2.1 we deal here with spaces $A^s_{pq}(\Omega)$ defined by restriction, including the special case $\mathcal{W}_p^k(\Omega)$ in the above context. But at least for Sobolev spaces the intrinsic version according to Definition 2.53 is natural and makes sense. It attracted a lot of attention since the days of Sobolev, [Sob50], including questions of type (2.226) and (2.242). The related standard reference nowadays is Maz'ya's book [Maz85].

Chapter 3
Spaces on thick domains

3.1 Thick domains

3.1.1 Introduction

Recall that open sets in $\mathbb{R}^n$ are denoted as domains. In Definition 2.1 we introduced in arbitrary domains Ω with $\Omega \neq \mathbb{R}^n$ the spaces $A^s_{pq}(\Omega)$ and $\tilde{A}^s_{pq}(\Omega)$ as subspaces of $D'(\Omega)$. So far we obtained in Theorem 2.36 common orthonormal u-wavelet bases for all spaces

$$L_p(\Omega) = F^0_{p,2}(\Omega) = \widetilde{F}^0_{p,2}(\Omega) = F^{0,\mathrm{rloc}}_{p,2}(\Omega), \quad 1 < p < \infty, \tag{3.1}$$

in arbitrary domains based on (2.113). On the one hand there is little hope to extend this assertion to other spaces $A^s_{pq}(\Omega)$ and $\tilde{A}^s_{pq}(\Omega)$. But on the other hand we obtained in Theorem 2.38 for the refined localisation spaces

$$F^{s,\mathrm{rloc}}_{pq}(\Omega), \quad 0 < p \leq \infty,\ 0 < q \leq \infty,\ s > \sigma_{pq}, \tag{3.2}$$

($q = \infty$ if $p = \infty$) in arbitrary domains Ω satisfactory orthonormal u-wavelet bases. As described in Remark 2.17 we have so far

$$F^{s,\mathrm{rloc}}_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega), \quad 0 < p, q \leq \infty,\ s > \sigma_{pq}, \tag{3.3}$$

($q = \infty$ if $p = \infty$) if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. It is our first aim to extend this assertion to E-thick domains which are more natural for questions of this type. For this purpose we introduce first several classes of domains which will play a crucial role in what follows.

3.1.2 Classes of domains

Tacitly we always assume that a domain Ω in $\mathbb{R}^n$ is not empty, $\Omega \neq \emptyset$. Let $l(Q)$ be the side-length of a (finite) cube Q in $\mathbb{R}^n$ with sides parallel to the axes of coordinates and let $|L|$ be the length of a rectifiable curve (closed path) L in $\mathbb{R}^n$. Let I be an (arbitrary) index set. Then

$$a_i \sim b_i \quad \text{for } i \in I \ \ \text{(equivalence)} \tag{3.4}$$

for two sets of positive numbers $\{a_i : i \in I\}$ and $\{b_i : i \in I\}$ means that there are two positive numbers c_1 and c_2 such that

$$c_1 a_i \leq b_i \leq c_2 a_i \quad \text{for all } i \in I. \tag{3.5}$$

Let $\mathrm{dist}(\Gamma^1, \Gamma^2)$ be as in (2.21).

Definition 3.1. Let Ω be a domain (= open set) in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and $\Gamma = \partial\Omega$.

(i) Then Ω is said to be an (ε, δ)-domain, $0 < \varepsilon < \infty$, $0 < \delta < \infty$, if it is connected and if for any $x \in \Omega$, $y \in \Omega$ with $|x - y| < \delta$ there is a curve $L \subset \Omega$, connecting x and y such that $|L| \le \varepsilon^{-1} |x - y|$ and

$$\operatorname{dist}(z, \Gamma) \ge \varepsilon \min(|x - z|, |y - z|), \quad z \in L. \tag{3.6}$$

(ii) Then Ω is said to be E-thick (exterior thick) if one finds for any interior cube $Q^i \subset \Omega$ with

$$l(Q^i) \sim 2^{-j}, \quad \operatorname{dist}(Q^i, \Gamma) \sim 2^{-j}, \quad j \ge j_0 \in \mathbb{N}, \tag{3.7}$$

a ***complementing*** exterior cube $Q^e \subset \Omega^c = \mathbb{R}^n \setminus \Omega$ with

$$l(Q^e) \sim 2^{-j}, \quad \operatorname{dist}(Q^e, \Gamma) \sim \operatorname{dist}(Q^i, Q^e) \sim 2^{-j}, \quad j \ge j_0 \in \mathbb{N}. \tag{3.8}$$

(iii) Then Ω is said to be I-thick (interior thick) if one finds for any exterior cube $Q^e \subset \Omega^c$ with

$$l(Q^e) \sim 2^{-j}, \quad \operatorname{dist}(Q^e, \Gamma) \sim 2^{-j}, \quad j \ge j_0 \in \mathbb{N}, \tag{3.9}$$

a ***reflected*** cube $Q^i \subset \Omega$ with

$$l(Q^i) \sim 2^{-j}, \quad \operatorname{dist}(Q^i, \Gamma) \sim \operatorname{dist}(Q^e, Q^i) \sim 2^{-j}, \quad j \ge j_0 \in \mathbb{N}. \tag{3.10}$$

(iv) Then Ω is said to be thick if it is both E-thick and I-thick.

Remark 3.2. The equivalence constants in (3.7)–(3.10) are independent of j. In other words, a domain Ω is called E-thick if for any choice of positive numbers c_1, c_2, c_3, c_4 and $j_0 \in \mathbb{N}$ there are positive numbers c_5, c_6, c_7, c_8 such that one finds for each interior cube $Q^i \subset \Omega$ with

$$c_1 2^{-j} \le l(Q^i) \le c_2 2^{-j}, \quad c_3 2^{-j} \le \operatorname{dist}(Q^i, \Gamma) \le c_4 2^{-j}, \tag{3.11}$$

$j \ge j_0$, an exterior cube $Q^e \subset \Omega^c$ with

$$c_5 2^{-j} \le l(Q^e) \le c_6 2^{-j}, \quad c_7 2^{-j} \le \operatorname{dist}(Q^e, \Gamma) \le \operatorname{dist}(Q^i, Q^e) \le c_8 2^{-j}, \tag{3.12}$$

$j \ge j_0$. One may think about the Whitney cubes according to (2.22), (2.23) identifying Q^0_{lr} or Q^1_{lr} with Q^i in (3.7) or (3.11) (where now $l = j$). One checks quite easily that Ω *is* E*-thick if, and only if, one has* (3.12) *for some positive* c_5, c_6, c_7, c_8 *for the indicated Whitney cubes* (*with related positive* c_2, c_2, c_3, c_4). Similarly one can detail (3.9), (3.10) in case of I-thick domains.

Remark 3.3. In this Chapter 3 we are mainly interested in E-thick domains which prove to be the natural class of domains in connection with wavelet bases in function spaces. In the next Chapter 4 we use these results to study the so-called extension problem. Then I-thick domains are coming in naturally. On the one hand by Proposition 3.6 below (ε,δ)-domains are special I-thick domains. On the other hand it is well known that (ε,δ)-domains in the plane $\mathbb{R}^2$ play not only a crucial role in the theory of quasi-conformal mappings in $\mathbb{R}^2$ but also in connection with the question of the extendability of the Sobolev spaces $W_p^k(\Omega)$ according to Definition 2.53 to $\mathbb{R}^2$, hence the problem of whether there exists a linear and bounded extension operator Ext as indicated in (2.226). What is meant by (ε,δ)-domains Ω in $\mathbb{R}^n$ can be seen in Figure 3.1. In particular (3.6) ensures that with L there is also a surrounding croissant-like

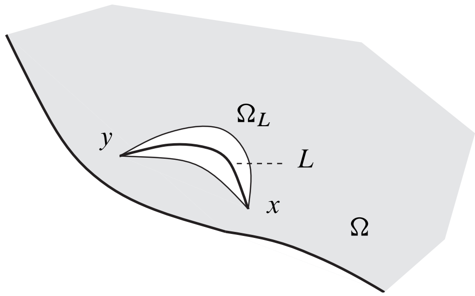

Figure 3.1

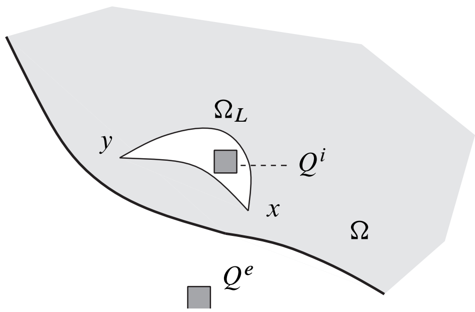

Figure 3.2

subdomain $\Omega_L \subset \Omega$. According to [Jon81] for any (ε,δ)-domain Ω in $\mathbb{R}^n$ there exists a linear and bounded extension operator

$$\mathrm{Ext}_p^k \colon W_p^k(\Omega) \hookrightarrow W_p^k(\mathbb{R}^n), \quad 1 \le p \le \infty,\ k \in \mathbb{N}, \tag{3.13}$$

with $\mathrm{Ext}_p^k f|\Omega = f$, where Definition 2.53 is naturally extended to $p=1$ and $p=\infty$. But it is not a common extension operator as needed in many applications. However the method in [Jon81] has been modified recently in [Rog06] such that one obtains common extension operators of type (3.13). The considerations in [Jon81] have been extended in [Chr84], [Miy93] to spaces of fractional order and in [See89], [Miy98] to the above spaces

$$F^s_{pq}(\Omega) \quad \text{with } 0 < p < \infty, 0 < q \le \infty, s > \sigma_{pq}, \tag{3.14}$$

according to Definition 2.1. Some assertions beyond (ε, δ)-domains may also be found in [Ry00]. In the plane $\mathbb{R}^2$ the property to be an (ε, δ)-domain is not only sufficient for the extendability of intrinsically defined Sobolev spaces but (with some mild additional assumptions) also necessary. The related literature may be found in [Jon81]. However the situation is not so simple and a detailed discussion may be found in [Maz85], Section 1.1.16, §1.5. Our intentions are different. According to Definition 2.1 all spaces considered in this book are introduced by restriction. In case of irregular domains this may change the situation substantially. We refer in this context to Remark 2.54. We return in Chapter 4 to the extension problem but in contrast to the above-mentioned literature in the framework of wavelet bases. First we complement the above notation about sets and classes of domains as follows.

Recall that balls in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ and of radius $r > 0$ are denoted as $B(x, r)$. Furthermore for $2 \le n \in \mathbb{N}$,

$$\mathbb{R}^{n-1} \ni x' \mapsto h(x') \in \mathbb{R} \tag{3.15}$$

is called a *Lipschitz function* (on $\mathbb{R}^{n-1}$) if there is a number $c > 0$ such that

$$|h(x') - h(y')| \le c\,|x' - y'| \quad \text{for all } x' \in \mathbb{R}^{n-1},\ y' \in \mathbb{R}^{n-1}. \tag{3.16}$$

Definition 3.4. (i) Let $n \in \mathbb{N}$. A closed set Γ in $\mathbb{R}^n$ is said to be porous if there is a number η with $0 < \eta < 1$ such that one finds for any ball $B(x, r)$ centred at $x \in \mathbb{R}^n$ and of radius r with $0 < r < 1$, a ball $B(y, \eta r)$ with

$$B(y, \eta r) \subset B(x, r) \quad \text{and} \quad B(y, \eta r) \cap \Gamma = \emptyset. \tag{3.17}$$

(ii) Let $2 \le n \in \mathbb{N}$. A special Lipschitz domain (C^∞ domain) in $\mathbb{R}^n$ is the collection of all points $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ such that

$$h(x') < x_n < \infty, \tag{3.18}$$

where $h(x')$ is a Lipschitz function according to (3.15), (3.16) (a bounded C^∞ function).

(iii) Let $2 \le n \in \mathbb{N}$. A bounded Lipschitz domain (C^∞ domain) in $\mathbb{R}^n$ is a bounded domain Ω in $\mathbb{R}^n$ where the boundary $\Gamma = \partial\Omega$ can be covered by finitely many open balls B_j in $\mathbb{R}^n$ with $j = 1, \ldots, J$, centred at Γ such that

$$B_j \cap \Omega = B_j \cap \Omega_j \quad \text{for } j = 1, \ldots, J, \tag{3.19}$$

where Ω_j are rotations of suitable special Lipschitz domains (C^∞ domains) in $\mathbb{R}^n$.

Remark 3.5. Of course we always assume that bounded (Lipschitz or C^∞) domains are not empty. Later one we need the porosity of the boundary $\Gamma = \partial\Omega$ of some E-thick domains in connection with spaces $A^0_{pq}(\Omega)$ of smoothness zero. A detailed discussion about porosity (also called *ball condition*) especially in connection with fractal measures may be found in [T01], Section 9.16–9.19, pp. 138–141. If one knows (3.17) only for balls $B(x,r)$ centred at $x \in \Gamma$, then one has (3.17) for all balls with $\eta/2$ in place of η, hence Γ is porous. In Definition 5.40 we introduce so-called *cellular domains*, a class of domains between Lipschitz and C^∞.

3.1.3 Properties and examples

We wish to compare the diverse types of domains introduced in the preceding Section 3.1.2 and to illustrate the situation by a few examples. Let Ω be a domain (= open set) in $\mathbb{R}^n$. Then $(\bar\Omega)^\circ$ is the largest domain ω with $\omega \subset \bar\Omega$. Of course, $\Omega \subset (\bar\Omega)^\circ$. We use the notation as introduced in Definitions 3.1, 3.4.

Proposition 3.6. (i) *Let Ω be an (ε,δ)-domain in $\mathbb{R}^n$. Then Ω is I-thick. Furthermore, $\partial\Omega$ is porous and $|\partial\Omega| = 0$.*

(ii) *Let $\Omega \neq \mathbb{R}^n$ be an arbitrary domain. Then one has the decomposition of $\mathbb{R}^n$ into three disjoint sets*

$$\mathbb{R}^n = \Omega \cup \partial\Omega \cup (\mathbb{R}^n\setminus\bar\Omega) \quad \textit{and} \quad \partial(\mathbb{R}^n\setminus\bar\Omega) \subset \partial\Omega. \tag{3.20}$$

Furthermore,

$$\partial\Omega = \partial(\mathbb{R}^n\setminus\bar\Omega) \quad \textit{if, and only if,} \quad (\bar\Omega)^\circ = \Omega. \tag{3.21}$$

(iii) *If Ω is an E-thick domain in $\mathbb{R}^n$ then $(\bar\Omega)^\circ = \Omega$ and $\mathbb{R}^n\setminus\bar\Omega$ is I-thick.*

(iv) *If Ω is an I-thick domain in $\mathbb{R}^n$ and $\bar\Omega \neq \mathbb{R}^n$, then $\mathbb{R}^n\setminus\bar\Omega$ is E-thick.*

Proof. Step 1. We prove (i). Let Ω be an (ε,δ)-domain and let Q^e be as in Figure 3.2, with (3.9) where $j_0 \in \mathbb{N}$ is assumed to be sufficiently large. Recall that Ω is connected. Then there is a point $x \in \Omega$ with

$$\operatorname{dist}(x,\partial\Omega) \sim \operatorname{dist}(x,Q^e) \sim 2^{-j} \tag{3.22}$$

and a second point $y \in \Omega$ with $|x-y| \geq 2^{-j}$. As indicated in Figure 3.2, one finds in the croissant-domain Ω_L a cube Q^i with the desired properties. This argument proves also the porosity of $\partial\Omega$. Then one obtains $|\partial\Omega| = 0$ from [T01], Sections 9.16, 9.17, pp. 138–39, where porosity was called ball condition.

Step 2. Sine $\bar\Omega = \Omega \cup \partial\Omega$ the decomposition in (3.20) is obvious. The second assertion in (3.20) follows from

$$\partial(\mathbb{R}^n\setminus\bar\Omega) \subset \bar\Omega \quad \text{and} \quad \partial(\mathbb{R}^n\setminus\bar\Omega) \cap \Omega = \emptyset.$$

If $x \in \partial\Omega$ for a domain Ω with $(\bar\Omega)^\circ = \Omega$ then $x \in \partial(\mathbb{R}^n\setminus\bar\Omega)$. Together with (3.20) one obtains $\partial\Omega = \partial(\mathbb{R}^n\setminus\bar\Omega)$. Conversely let $\partial\Omega = \partial(\mathbb{R}^n\setminus\bar\Omega)$ and $x \in (\bar\Omega)^\circ$. Then either $x \in \Omega$ or $x \in \partial\Omega = \partial(\mathbb{R}^n \setminus \overline{\Omega})$. But the latter is not possible. Hence $x \in \Omega$ and $(\overline{\Omega})^\circ = \Omega$. This proves (ii).

Step 3. If Ω is E-thick and $x \in \partial\Omega$ then it follows $x \in \partial(\mathbb{R}^n \setminus \overline{\Omega})$ and by (3.20) that $\partial\Omega = \partial(\mathbb{R}^n \setminus \overline{\Omega})$. Now one obtains $(\overline{\Omega})^\circ = \Omega$ by (3.21). Furthermore, $\mathbb{R}^n \setminus \overline{\Omega}$ is I-thick. This proves (iii). Part (iv) follows from Definition 3.1. □

Remark 3.7. The first assertion in part (i) of the above proposition is a cornerstone of the considerations in [Jon81], [See89] in connection with the extension problems (3.13), (3.14). In good agreement with Remark 3.2 the arguments in these papers rely on Whitney cubes with universal equivalence constants. But there exist connected I-thick domains which are not (ε, δ)-domains. Let $0 < \alpha < 1$ and let locally $\Omega \subset \mathbb{R}^2$ below the curve $x_2 = |x_1|^\alpha$, Figure 3.3. Then Ω is I-thick, but not an (ε, δ)-domain

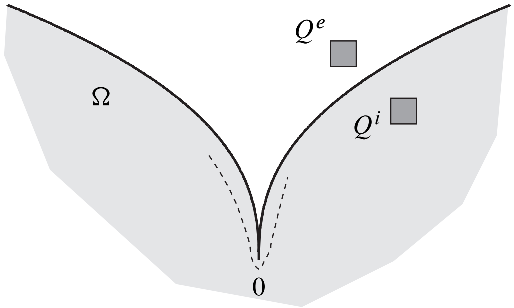

Figure 3.3

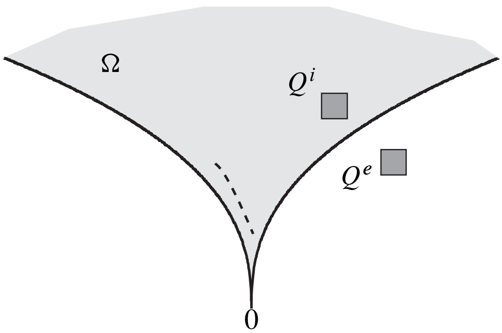

Figure 3.4

(the indicated connecting curves are too long). Furthermore if Ω is locally above the curve $x_2 = |x_1|^\alpha$ as indicated in Figure 3.4, then Ω is E-thick but not an (ε, δ)-domain (possible croissants are not fat enough). But there is an other difference between E-thick, I-thick domains on the one hand and C^∞, Lipschitz, (ε, δ)-domains on the other hand. There are E-thick domains Ω and I-thick domains Ω such that $|\partial\Omega| > 0$ with the consequences discussed so far in Remark 2.2. We give an example and return for this purpose to the construction (2.11), (2.12). Let I_l be the same intervals as there. Let J_l^0 with $l \in \mathbb{N}$ be intervals in $(0,1)$ of length $\varepsilon \cdot 2^{-l}$ and $\operatorname{dist}(J_l^0, 0) \sim 2^{-l}$. Similarly J_l^1 are corresponding intervals approaching 1. Let $\varepsilon > 0$ be small and

$$\Omega = \bigcup_{l=1}^{\infty} \big(I_l \cup J_l^0 \cup J_l^1 \big) . \tag{3.23}$$

Then Ω is I-thick with

$$\partial\Omega = [0,1] \backslash \Omega \quad \text{and} \quad |\partial\Omega| > 0. \tag{3.24}$$

To construct a corresponding E-thick domain we use the domain Ω in (2.11) now written as

$$\Omega = \bigcup_{l=1}^{\infty} I_l = \bigcup_{l=1}^{\infty} I_l^0, \tag{3.25}$$

where I_l^0 are disjoint intervals. We decompose each I_l^0 into

$$I_l^0 = I_l^1 \cup \{x_l^k\}_{k=1}^{\infty} \cup I_l^2 \tag{3.26}$$

where I_l^1 is the union of disjoint intervals $I_{l,k}^1$ of length $\sim 2^{-k} |I_l^0|$, $k \in \mathbb{N}$. Similarly for I_l^2. This can be done in such a way that I_l^1 is E-thick with the related exterior intervals in I_l^2 and vice versa. Then

$$\Omega^1 = \bigcup_{l=1}^{\infty} I_l^1 \quad \text{and} \quad \Omega^2 = \bigcup_{l=1}^{\infty} I_l^2 \tag{3.27}$$

are E-thick and

$$0 < |\partial\Omega| \le |\partial\Omega^1| + |\partial\Omega^2|. \tag{3.28}$$

Hence there are E-thick domains ω with $|\partial\omega| > 0$.

Thick domains, (ε,δ)-domains and bounded Lipschitz domains have the same meaning as in Definitions 3.1 and 3.4.

Proposition 3.8. (i) *Any bounded Lipschitz domain is thick.*

(ii) *Any connected bounded Lipschitz domain is an (ε,δ)-domain.*

(iii) *The classical snowflake domain Ω in the plane $\mathbb{R}^2$ according to Figure 3.5, is a thick (ε,δ)-domain.*

Proof. Parts (i) and (ii) are more or less obvious. If Ω is the indicated snowflake domain then it is also and (ε,δ)-domain as can be seen elementary. Then it follows by Proposition 3.6 (i) that Ω is I-thick. It remains to prove that Ω is also E-thick. This

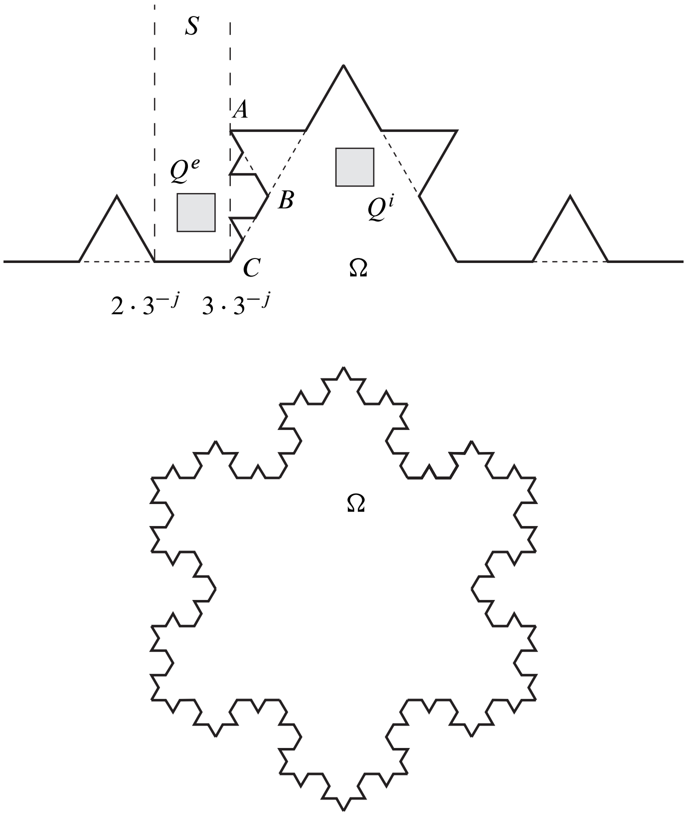

Figure 3.5.

follows from a careful examination of the indicated construction in Figure 3.5. Starting at level 3^{-j} it follows by geometrical reasoning that the following construction remains inside of the triangle ABC, whereas the indicated strip S, vertically translated by 3^{-j}, remains free of points of $\partial\Omega$. Hence for Q^i with $l(Q^i) \sim 3^{-j}$ as shown one finds in this strip a complementing cube Q^e with $l(Q^e) \sim 3^{-j}$. □

Remark 3.9. The examples in Remark 3.7 make clear that E-thick and I-thick domains may be rather bizarre. Even the more regular (ε, δ)-domains cover some domains with fractal boundary. In connection with the above snowflake domain we refer to [Maz85], Section 1.5.1, pp. 70–71, where one finds an alternative construction of a snowflake-like (ε, δ)-domain in the plane $\mathbb{R}^2$ based on squares instead of equilateral triangles. Otherwise we believe that (exterior and interior) thick domains are at the heart of the matter for problems of the theory of function spaces on domains as considered in this book: wavelet characterisations, linear extensions to $\mathbb{R}^n$, and intrinsic descriptions. However there is a large variety of similar notation adapted to (other) specific questions.

Based on [TrW96] and [ET96], Section 2.5, we describe in Remark 4.29 below what is meant by (*exterior* and *interior*) *regular domains* and what can be said about atomic representations for related function spaces in these domains. Furthermore we refer to the *corkscrew property* for *non-tangentially accessible domains* introduced in [JeK82] in order to study boundary value problems for second order elliptic equations. Details and generalisations (domains of *class* S) may be found in [Ken94], pp. 4, 8. In connection with Sobolev and Poincaré inequalities for function spaces (preferably Sobolev spaces) conditions of the above type play a role resulting in *John domains* and *plump domains*. Details, references, examples and discussions may be found in [EdE04], Section 4.3.

3.2 Wavelet bases in $\bar{A}^s_{pq}(\Omega)$

3.2.1 The spaces $\widetilde{F}^s_{pq}(\Omega)$

In arbitrary domains Ω in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ we have the orthonormal u-wavelet bases

$$\left\{\Phi^j_r :\ j \in \mathbb{N}_0;\ r = 1, \ldots, N_j\right\} \quad \text{with } N_j \in \bar{\mathbb{N}} \tag{3.29}$$

in $L_2(\Omega)$ according to Theorem 2.33 based on Definitions 2.4, 2.31. We extended this assertion in Theorem 2.38 to the refined localisation spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ with

$$0 < p \le \infty, \quad 0 < q \le \infty,\ s > \sigma_{pq}, \tag{3.30}$$

($q = \infty$ if $p = \infty$) as introduced in Definition 2.14 with (2.56)–(2.58). In particular, any $f \in F^{s,\mathrm{rloc}}_{pq}(\Omega)$ can be uniquely represented as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r\, 2^{-jn/2}\, \Phi^j_r, \quad \lambda \in f^s_{pq}(\mathbb{Z}_\Omega), \tag{3.31}$$

where $f^s_{pq}(\mathbb{Z}_\Omega)$ are the sequence spaces according to Definition 2.6 and

$$\lambda^j_r = \lambda^j_r(f) = 2^{jn/2}\left(f, \Phi^j_r\right) = 2^{jn/2} \int_\Omega f(x)\, \Phi^j_r(x)\, \mathrm{d}x. \tag{3.32}$$

It is the main aim of Section 3.2 to extend these wavelet representations (bases, isomorphisms onto related sequence spaces) to some spaces $\tilde{A}^s_{pq}(\Omega)$ and $A^s_{pq}(\Omega)$ according to Definition 2.1 where Ω is now assumed to be E-thick. First we prove by direct arguments that the spaces $\widetilde{F}^s_{pq}(\Omega)$ with (3.30) have the same wavelet representations (3.31), (3.32) as the spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ with the consequence that these two spaces coincide.

Proposition 3.10. *Let* Ω *be an* E*-thick domain in* $\mathbb{R}^n$ *according to Definition* 3.1 (ii). *Let*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s > \sigma_{pq}, \tag{3.33}$$

($q = \infty$ *if* $p = \infty$). *Let* $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ *be as in Definition* 2.14, (2.56)–(2.58), *and* $\widetilde{F}^s_{pq}(\Omega)$ *be as in Definition* 2.1 (ii) (*with* $\widetilde{F}^s_{\infty\infty}(\Omega) = \widetilde{B}^s_{\infty\infty}(\Omega)$). *Then*

$$F^{s,\mathrm{rloc}}_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega). \tag{3.34}$$

Proof. As indicated above we prove that $\widetilde{F}^s_{pq}(\Omega)$ has the same wavelet representation as $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ with (3.34) as a consequence. Let f be given by (3.31) where $\{\Phi^j_r\}$ is an u-wavelet system according to Definition 2.4 with $u > s$. After correct normalisations one has an atomic representation in $F^s_{pq}(\mathbb{R}^n)$ (no moment conditions are needed). Then it follows from Definition 2.6 and Theorem 1.7 that

$$\|f \,|\widetilde{F}^s_{pq}(\Omega)\| \le \|f \,|\widetilde{F}^s_{pq}(\bar{\Omega})\| = \|f \,|F^s_{pq}(\mathbb{R}^n)\| \le c\,\|\lambda \,|f^s_{pq}(\mathbb{Z}_\Omega)\|. \tag{3.35}$$

Conversely let $f \in \widetilde{F}^s_{pq}(\Omega)$. If $p < \infty$ then it follows from (2.71) with $1 < v < \infty$ and Theorem 2.36 that f can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r(f)\, 2^{-jn/2}\, \Phi^j_r \tag{3.36}$$

at least in $L_v(\Omega)$. If $p = \infty$ and, hence, also $v = \infty$, then one has (3.36) at least locally. But this is sufficient also in case of $p = q = \infty$ (recall that all spaces are considered as subspaces of $D'(\Omega)$). We wish to apply Theorem 1.15 (ii) with $A = 0$ and $u = B > s$ to the kernels

$$k^j_r = 2^{jn/2}\Phi^j_r, \quad j \in \mathbb{N}_0;\ r = 1, \ldots, N_j. \tag{3.37}$$

As for the correct normalisation we refer to Definition 2.4, based on (2.27) and (1.39). In case of the basic wavelets (2.32) and the interior wavelets (2.33) these kernels have the required moment conditions as a consequence of $u > s$ and (1.88). This need not to be the case for the boundary wavelets in Definition 2.4 (iii). Let k^j_r be given by (3.37) where Φ^j_r is a boundary wavelet according to (2.34), (2.35). We may assume that $j \ge j_0$ where j_0 is sufficiently large. Then

$$\operatorname{supp} k^j_r \subset Q^i \tag{3.38}$$

for a suitable interior cube with (3.7). Let Q^e be a related complementing cube with (3.8). Then there is a function $\mathring{k}^j_r \in C^u(\mathbb{R}^n)$ with $\operatorname{supp} \mathring{k}^j_r \subset Q^e$ such that $\tilde{k}^j_r$,

$$\tilde{k}^j_r(x) = k^j_r(x) + \mathring{k}^j_r(x), \quad x \in \mathbb{R}^n, \tag{3.39}$$

is an admitted kernel having in particular the required moment conditions. The existence of such complementing functions $\mathring{k}^j_r$ is quite plausible but not obvious. A detailed construction may be found in [TrW96], pp. 665–66. Let $g \in \widetilde{F}^s_{pq}(\bar{\Omega})$ with $g|\Omega = f$ and

$$\|g \,|F^s_{pq}(\mathbb{R}^n)\| = \|g \,|\widetilde{F}^s_{pq}(\bar{\Omega})\| \le 2\,\|f \,|\widetilde{F}^s_{pq}(\Omega)\|. \tag{3.40}$$

Since $\operatorname{supp} g \subset \bar{\Omega}$ one has

$$\int_{\mathbb{R}^n} \tilde{k}^j_r(x)\, g(x)\, \mathrm{d}x = \int_{\Omega} k^j_r(x)\, f(x)\, \mathrm{d}x. \tag{3.41}$$

Now we can apply Theorem 1.15. One obtains that

$$\|\lambda(f)\,|\,f^s_{pq}(\mathbb{Z}_\Omega)\| \le c\, \|g\,|\,F^s_{pq}(\mathbb{R}^n)\| \le c'\, \|f\,|\,\widetilde{F}^s_{pq}(\Omega)\|. \tag{3.42}$$

Together with (3.35) it follows that $\widetilde{F}^s_{pq}(\Omega)$ is the collection of all f which can be uniquely represented by (3.36) with

$$\|f\,|\,\widetilde{F}^s_{pq}(\Omega)\| \sim \|\lambda(f)\,|\,f^s_{pq}(\mathbb{Z}_\Omega)\|. \tag{3.43}$$

But according to Theorem 2.38 this is the same as for the corresponding representation in $F^{s,\mathrm{rloc}}_{pq}(\Omega)$. This proves (3.34). □

3.2.2 The spaces $\bar{A}^s_{pq}(\Omega)$ I

We are now ready for the main result of Chapter 3. Let $A^s_{pq}(\Omega)$ and $\tilde{A}^s_{pq}(\Omega)$ be the spaces introduced in Definition 2.1 for arbitrary domains Ω in $\mathbb{R}^n$ with $\Omega \ne \mathbb{R}^n$, considered as subspaces of $D'(\Omega)$. Let

$$\sigma_p = n\Big(\frac{1}{p} - 1\Big)_+ \quad \text{and} \quad \sigma_{pq} = n\Big(\frac{1}{\min(p,q)} - 1\Big)_+, \quad 0 < p,q \le \infty, \tag{3.44}$$

with $b_+ = \max(b,0)$ if $b \in \mathbb{R}$, be the same numbers as in (1.32).

Definition 3.11. Let Ω be an E-thick domain in $\mathbb{R}^n$ according to Definition 3.1 (ii). Then

$$\bar{F}^s_{pq}(\Omega) = \begin{cases} \widetilde{F}^s_{pq}(\Omega) & \text{if } 0 < p < \infty,\, 0 < q \le \infty,\, s > \sigma_{pq}, \\ F^0_{pq}(\Omega) & \text{if } 1 < p < \infty,\, 1 \le q < \infty,\, s = 0, \\ F^s_{pq}(\Omega) & \text{if } 0 < p < \infty,\, 0 < q \le \infty,\, s < 0, \end{cases} \tag{3.45}$$

and

$$\bar{B}^s_{pq}(\Omega) = \begin{cases} \widetilde{B}^s_{pq}(\Omega) & \text{if } 0 < p \le \infty,\, 0 < q \le \infty,\, s > \sigma_p, \\ B^0_{pq}(\Omega) & \text{if } 1 < p < \infty,\, 0 < q \le \infty,\, s = 0, \\ B^s_{pq}(\Omega) & \text{if } 0 < p \le \infty,\, 0 < q \le \infty,\, s < 0. \end{cases} \tag{3.46}$$

Remark 3.12. Recall that all spaces are considered as subspaces of $D'(\Omega)$. As always $\Omega \ne \mathbb{R}^n$. By Proposition 3.10 the refined localisation spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$, introduced in Definition 2.14 in arbitrary domains Ω in $\mathbb{R}^n$ with $\Omega \ne \mathbb{R}^n$, coincide with the spaces $\bar{F}^s_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega)$ if Ω is E-thick and $s > \sigma_{pq}$. This assertion has a little history mentioned briefly in Remark 2.17. In Definition 3.4 we recalled what is meant by bounded Lipschitz domains and bounded C^∞ domains in $\mathbb{R}^n$. In [T01], Theorem 5.14, p. 60–61, we proved (3.34) for bounded C^∞ domains in $\mathbb{R}^n$, called the *refined localisation property* for $\widetilde{F}^s_{pq}(\Omega)$. We obtained in [T06], Proposition 4.20, p. 208, the same assertion for $\widetilde{F}^s_{pq}(\Omega)$ in bounded Lipschitz domains under the additional restriction

$$1 < p, q \le \infty \ \ (q = \infty \text{ if } p = \infty), \quad s > 0.$$

This is now valid for all p, q, s in (3.33) since bounded Lipschitz domains are E-thick according to Proposition 3.8 (i). However in [T06], p. 209, Remark 4.21, footnote, we already indicated that such an extension might be possible. Now we have this refined localisation property of $\widetilde{F}^s_{pq}(\Omega)$ in all E-thick domains. This class of domains seems to be natural for questions of this type. In other words, beyond E-thick domains the refined localisation spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ according to (2.56)–(2.58) seem to be the right way to look at spaces of this type.

Let $b^s_{pq}(\mathbb{Z}_\Omega)$ and $f^s_{pq}(\mathbb{Z}_\Omega)$ be the same sequence spaces as in Definition 2.6. One word about the convergence of the series in the following theorem seems to be in order. The u-wavelet system $\{\Phi^j_r\}$ according to Definition 2.4 in Ω can also be considered as an u-wavelet system in $\mathbb{R}^n$. Hence instead of saying that a related series converges in $D'(\Omega)$ one could equally say that this series converges in $S'(\mathbb{R}^n)$. Similarly one can likewise say that such a series converges locally in $\bar{A}^\sigma_{pq}(\Omega)$ or locally in $A^\sigma_{pq}(\mathbb{R}^n)$. In particular, if Ω is bounded then convergence locally in $\bar{A}^\sigma_{pq}(\Omega)$ means convergence in $\bar{A}^\sigma_{pq}(\Omega)$. Hence as far as convergences and other technicalities are concerned (duality etc.) we are in the same position as in Theorem 1.20 where one finds corresponding comments in front of this theorem and in the first step of its proof.

Theorem 3.13. *Let Ω be an E-thick domain in $\mathbb{R}^n$ according to Definition 3.1* (ii). *Let for $u \in \mathbb{N}$,*

$$\left\{\Phi^j_r : \ j \in \mathbb{N}_0;\ r = 1, \dots, N_j\right\} \quad \text{with } N_j \in \bar{\mathbb{N}}, \tag{3.47}$$

be an orthonormal u-wavelet basis in $L_2(\Omega)$ according to Theorem 2.33 and Definitions 2.31, 2.4.

(i) *Let $\bar{F}^s_{pq}(\Omega)$ be the spaces in (3.45) and let*

$$u > \max(s, \sigma_{pq} - s), \quad s \ne 0. \tag{3.48}$$

Then $f \in D'(\Omega)$ is an element of $\bar{F}^s_{pq}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r \, 2^{-jn/2} \, \Phi^j_r, \quad \lambda \in f^s_{pq}(\mathbb{Z}_\Omega), \tag{3.49}$$

unconditional convergence being in $D'(\Omega)$ and locally in any space $\bar{F}^\sigma_{pq}(\Omega)$ with $\sigma < s$. Furthermore, if $f \in \bar{F}^s_{pq}(\Omega)$ then the representation (3.49) is unique with $\lambda = \lambda(f)$,

$$\lambda^j_r = \lambda^j_r(f) = 2^{jn/2} \left(f, \Phi^j_r\right), \tag{3.50}$$

and

$$I: f \mapsto \lambda(f) = \{2^{jn/2} (f, \Phi^j_r)\} \tag{3.51}$$

is an isomorphic map of $\bar{F}^s_{pq}(\Omega)$ *onto* $f^s_{pq}(\mathbb{Z}_\Omega)$. *If, in addition,* $q < \infty$ *then* $\{\Phi^j_r\}$ *is an unconditional basis in* $\bar{F}^s_{pq}(\Omega)$.

(ii) *Let* $\bar{B}^s_{pq}(\Omega)$ *be the spaces in* (3.46) *and let*

$$u > \max(s, \sigma_p - s), \quad s \neq 0. \tag{3.52}$$

Then $f \in D'(\Omega)$ *is an element of* $\bar{B}^s_{pq}(\Omega)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r \, 2^{-jn/2} \, \Phi^j_r, \quad \lambda \in b^s_{pq}(\mathbb{Z}_\Omega), \tag{3.53}$$

unconditional convergence being in $D'(\Omega)$ *and locally in any spaces* $\bar{B}^\sigma_{pq}(\Omega)$ *with* $\sigma < s$. *Furthermore, if* $f \in \bar{B}^s_{pq}(\Omega)$ *then the representation* (3.53) *is unique with* $\lambda = \lambda(f)$ *as in* (3.50) *and* I *in* (3.51) *is an isomorphic map of* $\bar{B}^s_{pq}(\Omega)$ *onto* $b^s_{pq}(\mathbb{Z}_\Omega)$. *If, in addition,* $p < \infty$, $q < \infty$, *then* $\{\Phi^j_r\}$ *is an unconditional basis in* $\bar{B}^s_{pq}(\Omega)$.

Proof. Step 1. Part (i) with $s > \sigma_{pq}$ is covered by Proposition 3.10. The proof given there applies also to the spaces $\bar{B}^s_{pq}(\Omega)$ with $s > \sigma_p$. Hence it remains to prove the theorem for the spaces with $s < 0$.

Step 2. We prove (i) for the third line in (3.45) in three steps. Let in particular

$$s < 0 \quad \text{and} \quad u > \sigma_{pq} - s. \tag{3.54}$$

Recall that the interior wavelets in (2.33) have the cancellation property (2.152). Then it follows by Theorem 1.7 (ii) that

$$a^j_r = 2^{-j(s-\frac{n}{p})} \, 2^{-jn/2} \, \Phi^j_r \tag{3.55}$$

are atoms in $F^s_{pq}(\mathbb{R}^n)$ if Φ^j_r are basic wavelets or interior wavelets according to Definition 2.4 (neglecting immaterial constants). As for the boundary wavelets Φ^j_r in (2.34), (2.35) we are in a similar situation as in (3.38), (3.39). By the same arguments one can complement

$$\Phi^j_r \text{ with } \operatorname{supp} \Phi^j_r \subset Q^i \quad \text{by} \quad \mathring{\Phi}^j_r \in C^u(\mathbb{R}^n) \text{ with } \operatorname{supp} \mathring{\Phi}^j_r \subset Q^e$$

such that a^j_r,

$$a^j_r(x) = 2^{-j(s-\frac{n}{p})} \, 2^{-jn/2} \left(\Phi^j_r(x) + \mathring{\Phi}^j_r(x)\right), \quad x \in \mathbb{R}^n, \tag{3.56}$$

is an atom in $F^s_{pq}(\mathbb{R}^n)$ having the needed moment conditions. Let now f be given by (3.49). Then

$$g = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} 2^{j(s-\frac{n}{p})} \, \lambda^j_r \, a^j_r, \quad \lambda \in f^s_{pq}(\mathbb{Z}_\Omega), \tag{3.57}$$

is an atomic decomposition of g in $F^s_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$. One has by Theorem 1.7 that

$$\| f \,|F^s_{pq}(\Omega)\| \le \| g \,|F^s_{pq}(\mathbb{R}^n)\| \le c\, \|\lambda \,|f^s_{pq}(\mathbb{Z}_\Omega)\|. \tag{3.58}$$

Step 3. Next we assume that $f \in F^s_{pq}(\Omega)$ can be represented by

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r(f)\, 2^{-jn/2}\, \Phi^j_r. \tag{3.59}$$

As in (3.36), (3.37) we interpret

$$\lambda^j_r(f) = 2^{jn/2}\, \big(f, \Phi^j_r\big) = \big(f, k^j_r\big), \quad k^j_r = 2^{jn/2}\, \Phi^j_r, \tag{3.60}$$

as local means with respect to the kernels k^j_r. Again the kernels k^j_r have the desired normalisation (1.39). One can apply Theorem 1.15 with $A = u > \sigma_{pq} - s$ and $B = 0$. In particular, no moment conditions are required. Then it follows that

$$\|\lambda(f) \,|f^s_{pq}(\mathbb{Z}_\Omega)\| \le c\, \| f \,|F^s_{pq}(\mathbb{R}^n)\|. \tag{3.61}$$

Let $g \in F^s_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$. Then $\lambda(f) = \lambda(g)$ and one has (3.61) with g in place of f on the right-hand side. Taking the infimum over all such g one arrives at (3.61) with Ω on the right-hand side instead of $\mathbb{R}^n$. Together with (3.58) one obtains

$$\|\lambda(f) \,|f^s_{pq}(\mathbb{Z}_\Omega)\| \sim \| f \,|F^s_{pq}(\Omega)\| \tag{3.62}$$

for all $f \in F^s_{pq}(\Omega)$ which can be represented by (3.59).

Step 4. In other words, it remains to prove that any $f \in F^s_{pq}(\Omega)$ can be represented by (3.59). Let $1 < p < \infty$ and $0 < q < \infty$. Then $S(\Omega) = S(\mathbb{R}^n)|\Omega$ is dense in $F^s_{pq}(\Omega)$. Since

$$S(\Omega) \subset L_p(\Omega) \hookrightarrow F^s_{pq}(\Omega) \tag{3.63}$$

it follows that $L_p(\Omega)$ is also dense in $F^s_{pq}(\Omega)$. According to Theorem 2.36 any $f \in L_p(\Omega)$ can be represented by (3.59). Let $f \in F^s_{pq}(\Omega)$ and let $\{f_l\} \subset L_p(\Omega)$ be an approximating sequence. By (3.62) with f_l in place of f and Fatou's lemma one has $\lambda(f) \in f^s_{pq}(\mathbb{Z}_\Omega)$. Then it follows from Step 2 that

$$g = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r(f)\, 2^{-jn/2}\, \Phi^j_r \in F^s_{pq}(\Omega). \tag{3.64}$$

By the same arguments as in $\mathbb{R}^n$ one obtains $f = g$, where details may be found in [T06], p. 155. This proves that any $f \in F^s_{pq}(\Omega)$ can be represented by (3.59). By

$$F^s_{p\infty}(\Omega) \hookrightarrow F^{s-\varepsilon}_{pp}(\Omega), \quad \varepsilon > 0, \tag{3.65}$$

one has a corresponding assertion for $F^s_{p\infty}(\Omega)$ with $1 < p < \infty$, $s < 0$. If $p \le 1$ then one obtains the desired representability from the continuous embedding

$$F^s_{pq}(\Omega) \hookrightarrow F^\sigma_{rr}(\Omega), \quad s - \tfrac{n}{p} = \sigma - \tfrac{n}{r},\ 1 < r < \infty. \tag{3.66}$$

This covers all cases and proves part (i) of the above theorem.

Step 5. The proof of part (ii) for the B-spaces is the same at least as long as $p < \infty$. Let $p = \infty$. If Ω is bounded then the representability (3.59) follows from

$$B^s_{\infty q}(\Omega) \hookrightarrow B^s_{pq}(\Omega), \quad 0 < p < \infty. \tag{3.67}$$

The question of the representability (3.59) in $D'(\Omega)$ is a local matter (it must be tested against $\varphi \in D(\Omega)$). Then it follows from the local nature of Φ^j_r and the above considerations that $f \in B^s_{\infty q}(\Omega)$ can also be represented by (3.59) if Ω is unbounded.

Step 6. By the above arguments it follows that I is the indicated isomorphism. In particular if $p < \infty$, $q < \infty$ then $\{\Phi^j_r\}$ is an unconditional basis. □

Remark 3.14. So far we have a satisfactory assertion for the spaces $A^0_{pq}(\Omega)$ of smoothness zero only in case of $L_p(\Omega) = F^0_{p,2}(\Omega)$, $1 < p < \infty$. We refer to Theorem 2.28. We return to spaces of smoothness zero in Section 3.2.4. Then it will be clear that one needs some further properties of the underlying domain Ω to extend the above theorem to spaces of smoothness zero.

3.2.3 Complemented subspaces

Theorem 3.13 gives the possibility to interpret $\tilde{A}^s_{pq}(\Omega)$ with $A = B$ or $A = F$ as a complemented subspace of $A^s_{pq}(\mathbb{R}^n)$. This will be of some use later on in Chapter 4 in connection with the extension problem.

As usual $T : A \hookrightarrow B$ means that T is a linear and bounded (continuous) operator (map) from the quasi-Banach space A into the quasi-Banach space B. If $T = \mathrm{id}$ is a linear and bounded embedding then we often shorten $\mathrm{id} : A \hookrightarrow B$ by $A \hookrightarrow B$.

A subspace B of a quasi-Banach space A is called *complemented* if there is a linear and bounded operator P, called a *projection*, with

$$P : A \hookrightarrow A, \quad PA = B, \quad P^2 = P. \tag{3.68}$$

If $b \in B$ and $b = Pa$ for some $a \in A$, then

$$Pb = P^2 a = Pa = b, \quad b \in B. \tag{3.69}$$

Hence P, restricted to B, is the identity and $B = \{a \in A : Pa = a\}$. In particular B is a closed subspace of A. This extends naturally well-known properties of projections in Banach spaces to quasi-Banach spaces.

The spaces $\widetilde{F}^s_{pq}(\Omega)$ with $s > \sigma_{pq}$ and $\widetilde{B}^s_{pq}(\Omega)$ with $s > \sigma_p$ are subspaces of $D'(\Omega)$. If Ω is E-thick then one has the unique representations (3.49), (3.53) and the isomorphic map I in (3.51) onto corresponding sequence spaces. In the proof of Proposition 3.10 and in Step 1 of the proof of Theorem 3.13 we considered (3.36) as an atomic representation in $\mathbb{R}^n$ (after correct normalisation of the $\mathbb{R}^n$-atoms Φ^j_r, where no moment conditions are needed). In particular,

$$\|f \,|\widetilde{F}^s_{pq}(\Omega)\| \sim \|f \,|F^s_{pq}(\mathbb{R}^n)\|, \quad f \in \widetilde{F}^s_{pq}(\Omega), \tag{3.70}$$

represented by (3.36), with p, q, s as in (3.33). Similarly for $\widetilde{B}^s_{pq}(\Omega)$. Let temporarily $\mathrm{id}\tilde{A}^s_{pq}(\Omega)$ be the spaces $\tilde{A}^s_{pq}(\Omega)$ interpreted in this way as closed subspaces of $A^s_{pq}(\mathbb{R}^n)$. Recall that $\tilde{A}^s_{pq}(\bar{\Omega})$ has been introduced in Definition 2.1 as a closed subspace of $A^s_{pq}(\mathbb{R}^n)$.

Proposition 3.15. *Let Ω be an E-thick domain in $\mathbb{R}^n$ according to Definition* 3.1 (ii). *Let $\tilde{A}^s_{pq}(\bar{\Omega})$ and $\tilde{A}^s_{pq}(\Omega)$ be the spaces introduced in Definition* 2.1 (ii). *Let $\mathrm{id}\tilde{A}^s_{pq}(\Omega)$ be as above.*

(i) *Let $0 < p < \infty$, $0 < q \le \infty$, $s > \sigma_{pq}$. Then $\mathrm{id}\widetilde{F}^s_{pq}(\Omega)$ is a complemented subspace both of $F^s_{pq}(\mathbb{R}^n)$ and of $\widetilde{F}^s_{pq}(\bar{\Omega})$. Furthermore,*

$$\widetilde{F}^s_{pq}(\bar{\Omega}) = \mathrm{id}\widetilde{F}^s_{pq}(\Omega) \oplus \{h \in F^s_{pq}(\mathbb{R}^n) : \operatorname{supp} h \subset \partial\Omega\}. \tag{3.71}$$

(ii) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s > \sigma_p$. Then $\mathrm{id}\widetilde{B}^s_{pq}(\Omega)$ is a complemented subspace both of $B^s_{pq}(\mathbb{R}^n)$ and of $\widetilde{B}^s_{pq}(\bar{\Omega})$. Furthermore,*

$$\widetilde{B}^s_{pq}(\bar{\Omega}) = \mathrm{id}\widetilde{B}^s_{pq}(\Omega) \oplus \{h \in B^s_{pq}(\mathbb{R}^n) : \ \operatorname{supp} h \subset \partial\Omega\}. \tag{3.72}$$

Proof. Let $f \in \widetilde{F}^s_{pq}(\Omega)$ with p, q, s as in part (i) be given by (3.36) now considered as an atomic representation in $F^s_{pq}(\mathbb{R}^n)$. If Φ^j_r are boundary wavelets then the related kernels $\tilde{k}^j_r$ have the same meaning as in (3.39), (3.38), complemented by

$$\tilde{k}^j_r = 2^{jn/2}\,\Phi^j_r, \quad j \in \mathbb{N}_0, \tag{3.73}$$

for basic and interior wavelets Φ^j_r. In analogy to (3.36) with (3.50) we put

$$Pf = \sum_{j=0}^{\infty}\sum_{r=1}^{N_j} \Lambda^j_r(f)\, 2^{-jn/2}\,\Phi^j_r, \quad f \in F^s_{pq}(\mathbb{R}^n), \tag{3.74}$$

with

$$\Lambda^j_r(f) = 2^{jn/2} \int_{\mathbb{R}^n} \tilde{k}^j_r(x)\, f(x)\,\mathrm{d}x. \tag{3.75}$$

On the one hand, (3.74) is an atomic decomposition in $F^s_{pq}(\mathbb{R}^n)$ to which Theorem 1.7 can be applied. On the other hand, (3.75) are local means in $F^s_{pq}(\mathbb{R}^n)$ having the necessary moment conditions according to Theorem 1.15. Both together result in

$$\|Pf \,|F^s_{pq}(\mathbb{R}^n)\| \le c\,\|\Lambda(f)\,|f^s_{pq}(\mathbb{Z}_\Omega)\| \le c'\|f\,|F^s_{pq}(\mathbb{R}^n)\|. \tag{3.76}$$

Hence, P is a linear and bounded operator in $F^s_{pq}(\mathbb{R}^n)$. Furthermore, $Pf \in \mathrm{id}\widetilde{F}^s_{pq}(\Omega)$. If $f \in \mathrm{id}\widetilde{F}^s_{pq}(\Omega)$ then (3.74) coincides with (3.59), (3.60). Hence $Pf = f$. In particular one has

$$P^2 = P \text{ in } F^s_{pq}(\mathbb{R}^n), \quad PF^s_{pq}(\mathbb{R}^n) = \mathrm{id}\widetilde{F}^s_{pq}(\Omega), \tag{3.77}$$

and (3.70). Hence P is a projection of $F^s_{pq}(\mathbb{R}^n)$ onto $\mathrm{id}\,\widetilde{F}^s_{pq}(\Omega)$. Furthermore, one has

$$P\{h \in F^s_{pq}(\mathbb{R}^n) : \operatorname{supp} h \subset \partial\Omega\} = 0 \tag{3.78}$$

and the decomposition (3.71). This proves part (i). The proof of part (ii) is the same. □

3.2.4 Porosity and smoothness zero

On the one hand Definition 3.11 applies also to some spaces $A^0_{pq}(\Omega)$ of smoothness zero in E-thick domains. On the other hand in Theorem 3.13 we excluded just these spaces. With exception of the spaces $L_p(\Omega)$, $1 < p < \infty$, where we have the satisfactory Theorem 2.36, spaces of smoothness zero require always some extra considerations. We need some fractal arguments, characteristic functions as pointwise multipliers and some more or less sophisticated interpolation. But nothing of this will be treated in this book in detail and for its own sake. So we will be brief, restricting ourselves to formulations, outline of proofs and references. First we complement Definition 3.1 (ii) where we introduced *E-thick domains* and Definition 3.4 (i) where we said what is meant by *porosity*. For this purpose we recall that a locally finite positive Radon measure μ in $\mathbb{R}^n$ is called *isotropic* if there is a continuous strictly increasing function h on the interval $[0, 1]$ with $h(0) = 0$, $h(1) = 1$, and

$$\mu\,(B(\gamma, r)) \sim h(r) \quad \text{with } \gamma \in \Gamma = \operatorname{supp}\mu,\ 0 < r < 1, \tag{3.79}$$

(where the equivalence constants according to (3.5) are independent of γ and r). For details, discussions, and conditions ensuring that Radon measures with these properties exist we refer to [T06], Section 1.15.1, pp. 95–97 (where we assumed that $\Gamma = \operatorname{supp}\mu$ is compact, but this is immaterial). There one finds also references and examples with

$$h(r) = r^d, \quad 0 \le r \le 1,\ 0 < d < n \tag{3.80}$$

as the most distinguished case (d-sets). We call a closed set Γ in $\mathbb{R}^n$ *uniformly porous* if it is porous according to Definition 3.4 (i) and if there is an isotropic Radon measure μ with (3.79).

Definition 3.16. Let Ω be a domain (= open set) in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $\Gamma = \partial\Omega$.

(i) Then Ω is called E-porous if there is a number η with $0 < \eta < 1$ such that one finds for any ball $B(\gamma, r)$, centred at $\gamma \in \Gamma$ and of radius r with $0 < r < 1$, a ball $B(y, \eta r)$ with

$$B(y, \eta r) \subset B(\gamma, r) \quad \text{and} \quad B(y, \eta r) \cap \bar{\Omega} = \emptyset. \tag{3.81}$$

(ii) Then Ω is called uniformly E-porous if it is E-porous and if Γ is uniformly porous.

Remark 3.17. Porous sets (sometimes also called sets satisfying the ball condition) as introduced in Definition 3.4 (i) play a role in fractal geometry and also in the theory of function spaces. We refer to [T06], Definition 9.20, Remark 9.21, p. 393, Remark 2.32, p. 146, and [T01], Sections 9.16–9.18, pp. 138–141, where one finds also some properties. In particular as also mentioned in Remark 3.5, from (3.81) one has (3.17) with, say, $\eta/2$ in place of η. Hence the boundary Γ of an E-porous domain is porous. Although porosity and fractal measures are not the subject of this book we collect a few properties which are related to our intentions. In particular we wish to say a word how E-porous and E-thick domains introduced in Definition 3.1 are related to each other.

Proposition 3.18. *Let Ω be a domain ($=$ open set) in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and $\Gamma = \partial\Omega$.*

(i) *If Ω is E-porous then Ω is E-thick and $|\Gamma| = 0$.*

(ii) *If Ω is E-thick and if there is an isotropic Radon measure μ with (3.79) and*

$$h(2^{-j}) \le c\, 2^{(n-\lambda)l}\, h(2^{-j-l}) \quad \text{for all } j \in \mathbb{N}_0,\ l \in \mathbb{N}_0, \tag{3.82}$$

and some positive numbers c and λ, then Ω is uniformly E-porous.

(iii) *If Ω is E-thick and if there is an isotropic Radon measure μ with (3.79), (3.80) (which means that Γ is an d-set), then Ω is uniformly E-porous and $n-1 \le d < n$.*

Proof. Step 1. Part (i) follows easily from Definition 3.1 (ii) and [T01], Section 9.17, pp. 138–39 (as far as $|\Gamma| = 0$ is concerned).

Step 2. We prove (ii). According to [T01], Proposition 9.18, pp. 139–40 , (3.82) is a (necessary and sufficient) criterion for the porosity of Γ with (3.79). Since we assumed that Ω is E-thick it follows that Ω is uniformly E-porous.

Step 3. We prove (iii). If Γ is an d-set with $d < n$ then we have (3.82) with $\lambda = n - d$. By (ii) it follows that Ω is uniformly E-porous. It remains to prove $0 < n-1 \le d < n$. Since Ω is E-thick we may assume without restriction of generality that

$$Q_{n-1} = \{x = (x', 0) :\ 0 \le x_j \le 1;\ j = 1, \ldots, n-1\} \subset \Omega \tag{3.83}$$

and that on each line $\{x', x_n\}$ with $x' \in Q_{n-1}$ and $0 < x_n < \infty$ there is at least one point $\gamma \in \Gamma$. To avoid a contradiction it follows by elementary reasoning that $d \ge n-1$. □

Let $A^s_{pq}(\mathbb{R}^n)$ be one of the spaces according to Definition 1.1. Then $m \in L_\infty(\mathbb{R}^n)$ is said to be a *pointwise multiplier* for $A^s_{pq}(\mathbb{R}^n)$ if

$$f \mapsto mf \quad \text{generates a bounded map in } A^s_{pq}(\mathbb{R}^n). \tag{3.84}$$

Of course one has to say what this multiplication means. But this is not our topic. A careful discussion may be found in [RuS96], Section 4.2. We are interested only in characteristic functions χ_Ω of domains Ω as pointwise multipliers. This is a distinguished case which attracted a lot of attention. One may consult [RuS96], [T06], Section 2.3, and the literature mentioned there. We rely here especially on [Tri03]. Let $A^s_{pq}(\Omega)$ and $\tilde{A}^s_{pq}(\Omega)$ be the same spaces as in Definition 2.1.

Proposition 3.19. *Let Ω be a domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $\Gamma = \partial\Omega$ be uniformly porous* (*which means that Γ is porous according to Definition* 3.4 (i) *and that there is a Radon measure μ with* (3.79)). *Let χ_Ω be the characteristic function of Ω. Then there is a number δ with $0 < \delta < 1$ such that χ_Ω is a pointwise multiplier in*

$$F^s_{pq}(\mathbb{R}^n), \quad 1 < p < \infty,\ 1 \le q \le \infty,\ \delta\big(\tfrac{1}{p} - 1\big) < s < \tfrac{\delta}{p} \tag{3.85}$$

and in

$$B^s_{pq}(\mathbb{R}^n), \quad 1 < p < \infty,\ 0 < q \le \infty,\ \delta\big(\tfrac{1}{p} - 1\big) < s < \tfrac{\delta}{p}. \tag{3.86}$$

Furthermore,

$$F^s_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega) \tag{3.87}$$

with p, q, s as in (3.85), *and*

$$B^s_{pq}(\Omega) = \widetilde{B}^s_{pq}(\Omega) \tag{3.88}$$

with p, q, s as in (3.86).

Proof. The multiplier assertions are essentially covered by [Tri03], Corollaries 3, 4, where one may choose $\delta = \lambda$ with λ as in (3.82). (There we assumed that Ω is bounded, but this is immaterial.) In particular, ext,

$$(\operatorname{ext} f)(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \backslash \Omega, \end{cases} \tag{3.89}$$

is a linear and bounded extension operator from $F^s_{pq}(\Omega)$ with p, q, s as in (3.85) into $F^s_{pq}(\mathbb{R}^n)$ and from $B^s_{pq}(\Omega)$ with p, q, s as in (3.86) into $B^s_{pq}(\mathbb{R}^n)$. Then one obtains (3.87) and (3.88). □

Remark 3.20. Let $A^s_{pq}(\mathbb{R}^n)$ be either $F^s_{pq}(\mathbb{R}^n)$ in (3.85) or $B^s_{pq}(\mathbb{R}^n)$ in (3.86). Then one has as a by-product

$$\{h \in A^s_{pq}(\mathbb{R}^n) : \operatorname{supp} h \subset \partial\Omega\} = \{0\}. \tag{3.90}$$

Otherwise we refer for technicalities, more general assertions and the related literature to [RuS96], [T06] and in particular to [Tri03].

Now one can complement Theorem 3.13 by spaces of smoothness zero. All notation have the same meaning as there. This applies also to diverse technicalities.

Proposition 3.21. *Let Ω be an uniformly E-porous domain in $\mathbb{R}^n$ according to Definition* 3.16 (ii) *with $\Omega \neq \mathbb{R}^n$. Let for $u \in \mathbb{N}$,*

$$\big\{\Phi^j_r : j \in \mathbb{N}_0;\ r = 1, \ldots, N_j\big\} \quad \textit{with } N_j \in \bar{\mathbb{N}}, \tag{3.91}$$

be an orthonormal u-wavelet basis in $L_2(\Omega)$ according to Theorem 2.33 *and Definitions* 2.31, 2.4.

(i) *Then* $f \in D'(\Omega)$ *is an element of*

$$F^0_{pq}(\Omega), \quad 1 < p < \infty,\ 1 \le q < \infty, \tag{3.92}$$

if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r \, 2^{-jn/2} \, \Phi^j_r, \quad \lambda \in f^0_{pq}(\mathbb{Z}_\Omega), \tag{3.93}$$

unconditional convergence being in $D'(\Omega)$. *Furthermore, if* $f \in F^0_{pq}(\Omega)$ *then the representation* (3.93) *is unique with* $\lambda = \lambda(f)$,

$$\lambda^j_r = \lambda^j_r(f) = 2^{jn/2} \left(f, \Phi^j_r\right) \tag{3.94}$$

and

$$I : f \mapsto \lambda(f) = \left\{2^{jn/2} \left(f, \Phi^j_r\right)\right\} \tag{3.95}$$

is an isomorphic map of $F^0_{pq}(\Omega)$ *onto* $f^0_{pq}(\mathbb{Z}_\Omega)$. *Furthermore,* $\left\{\Phi^j_r\right\}$ *is an unconditional basis in* $F^0_{pq}(\Omega)$.

(ii) *Then* $f \in D'(\Omega)$ *is an element of*

$$B^0_{pq}(\Omega), \quad 1 < p < \infty,\ 0 < q \le \infty, \tag{3.96}$$

if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r \, 2^{-jn/2} \, \Phi^j_r, \quad \lambda \in b^0_{pq}(\mathbb{Z}_\Omega), \tag{3.97}$$

unconditional convergence being in $D'(\Omega)$. *Furthermore, if* $f \in B^0_{pq}(\Omega)$ *then the representation* (3.97) *is unique with* $\lambda = \lambda(f)$ *as in* (3.94) *and* I *in* (3.95) *is an isomorphic map of* $B^0_{pq}(\Omega)$ *onto* $b^0_{pq}(\mathbb{Z}_\Omega)$. *If, in addition,* $q < \infty$ *then* $\left\{\Phi^j_r\right\}$ *is an unconditional basis in* $B^0_{pq}(\Omega)$.

Proof. By Proposition 3.18 (i) the domain Ω is E-thick. Then one can apply Theorem 3.13 to the spaces $F^s_{pq}(\Omega)$ covered by (3.85), (3.87) with $s \ne 0$ (recall that $\delta < 1$). If $s > 0$ is chosen sufficiently small such that ext in (3.89) is an extension operator both for $F^s_{pq}(\Omega)$ and $F^{-s}_{pq}(\Omega)$ then the complex interpolation

$$\left[F^s_{pq}(\Omega), F^{-s}_{pq}(\Omega)\right]_{\frac{1}{2}} = F^0_{pq}(\Omega) \tag{3.98}$$

follows from the well-known $\mathbb{R}^n$-counterpart

$$\left[F^s_{pq}(\mathbb{R}^n), F^{-s}_{pq}(\mathbb{R}^n)\right]_{\frac{1}{2}} = F^0_{pq}(\mathbb{R}^n), \tag{3.99}$$

where $1 < p < \infty$, $1 \le q < \infty$. By real interpolation one obtains

$$\left(B^s_{pq}(\Omega), B^{-s}_{pq}(\Omega)\right)_{\frac{1}{2},q} = B^0_{pq}(\Omega) \tag{3.100}$$

where $1 < p < \infty$, $0 < q \le \infty$. There are counterparts for the corresponding sequence spaces $f^s_{pq}(\mathbb{Z}_\Omega)$ and $b^s_{pq}(\mathbb{Z}_\Omega)$. As far as complex and real interpolation and applications of common extension operators are concerned we refer to [T06], Section 1.11.8, pp. 69–72, where we dealt with problems of this type in case of bounded Lipschitz domains. A few comments about interpolation may also be found in Section 4.3.1 below. The arguments in the Step 2 of the proof of [T06], Theorem 4.22, p. 213, show how one can incorporate the desired interpolation formulas for the related sequence spaces. We used this type of argument also in connection with the proof of [T06], Corollary 4.28, pp. 216–17. Now the isomorphic mapping properties of I extend from $0 < |s|$ small to $s = 0$. This completes the proof. □

Remark 3.22. The proof of Theorem 3.13 relies on the observation that, according to Theorem 1.7, one does not need moment conditions for atoms if $s > \sigma_p$ or $s > \sigma_{pq}$, and that, according to Theorem 1.15, one does not need moment conditions for kernels of local means if $s < 0$. This excludes in particular spaces with $s = 0$. The above considerations might be considered as an attempt to circumvent this handicap. One has to pay a price: We strengthened the natural class of E-thick domains in Theorem 3.13 by the smaller class of uniformly E-porous domains in Proposition 3.21. On the one hand the notation of porosity (also called ball condition) is closely related to some problems in the theory of function spaces, especially to the question under which conditions the characteristic function χ_Ω of a domain Ω is a pointwise multiplier in spaces of smoothness zero. We refer to [T06], Remark 2.32, p. 146, and in particular to [Tri02], [Tri03]. On the other hand one may ask whether (first) moment conditions for atoms or kernels of local means for spaces of smoothness zero are really necessary. But this is the case. It came out quite recently that (first) moment conditions for atoms in spaces of smoothness zero are indispensable, even for $L_2(\mathbb{R}^n)$. We refer to [Vyb08a]. Corresponding assertions for spaces with $0 < p \le 1$ and $s = n\big(\frac{1}{p} - 1\big)$ may be found in [Sch08].

3.2.5 The spaces $\bar{A}^s_{pq}(\Omega)$ II

It seems to be reasonable to summarise the above considerations and to clip together Theorem 3.13 and Proposition 3.21. We do not repeat the technicalities preceding Theorem 3.13. Let $b^s_{pq}(\mathbb{Z}_\Omega)$ and $f^s_{pq}(\mathbb{Z}_\Omega)$ be the same sequence spaces as in Definition 2.6.

Theorem 3.23. *Let Ω be an uniformly E-porous domain in $\mathbb{R}^n$ according to Definition 3.16* (ii) *with $\Omega \neq \mathbb{R}^n$. Let for $u \in \mathbb{N}$,*

$$\big\{\Phi^j_r : j \in \mathbb{N}_0;\ r = 1, \dots, N_j\big\} \quad \text{with } N_j \in \bar{\mathbb{N}}, \tag{3.101}$$

be an orthonormal u-wavelet basis in $L_2(\Omega)$ according to Theorem 2.33 and Definitions 2.31, 2.4.

(i) *Let* $\bar{F}^s_{pq}(\Omega)$ *be the spaces in* (3.45) *and let*

$$u > \max(s, \sigma_{pq} - s). \tag{3.102}$$

Then $f \in D'(\Omega)$ *is an element of* $\bar{F}^s_{pq}(\Omega)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r \, 2^{-jn/2} \, \Phi^j_r, \quad \lambda \in f^s_{pq}(\mathbb{Z}_\Omega), \tag{3.103}$$

unconditional convergence being in $D'(\Omega)$ *and locally in any space* $\bar{F}^\sigma_{pq}(\Omega)$ *with* $\sigma < s$. *Furthermore, if* $f \in \bar{F}^s_{pq}(\Omega)$ *then the representation* (3.103) *is unique with* $\lambda = \lambda(f)$,

$$\lambda^j_r = \lambda^j_r(f) = 2^{jn/2} \left(f, \Phi^j_r \right), \tag{3.104}$$

and

$$I : f \mapsto \lambda(f) = \left\{ 2^{jn/2} \left(f, \Phi^j_r \right) \right\} \tag{3.105}$$

is an isomorphic map of $\bar{F}^s_{pq}(\Omega)$ *onto* $f^s_{pq}(\mathbb{Z}_\Omega)$. *If, in addition,* $q < \infty$ *then* $\left\{ \Phi^j_r \right\}$ *is an unconditional basis in* $\bar{F}^s_{pq}(\Omega)$.

(ii) *Let* $\bar{B}^s_{pq}(\Omega)$ *be the spaces in* (3.46) *and let*

$$u > \max(s, \sigma_p - s). \tag{3.106}$$

Then $f \in D'(\Omega)$ *is an element of* $\bar{B}^s_{pq}(\Omega)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r \, 2^{-jn/2} \, \Phi^j_r, \quad \lambda \in b^s_{pq}(\mathbb{Z}_\Omega), \tag{3.107}$$

unconditional convergence being in $D'(\Omega)$ *and locally in any space* $\bar{B}^\sigma_{pq}(\Omega)$ *with* $\sigma < s$. *Furthermore, if* $f \in \bar{B}^s_{pq}(\Omega)$ *then the representation* (3.107) *is unique with* $\lambda = \lambda(f)$ *as in* (3.104), *and* I *in* (3.105) *is an isomorphic map of* $\bar{B}^s_{pq}(\Omega)$ *onto* $b^s_{pq}(\mathbb{Z}_\Omega)$. *If, in addition,* $p < \infty$, $q < \infty$, *then* $\left\{ \Phi^j_r \right\}$ *is an unconditional basis in* $\bar{B}^s_{pq}(\Omega)$.

Proof. By Proposition 3.18 the domain Ω is E-thick. Then the above assertion follows from Theorem 3.13 and Proposition 3.21. □

Remark 3.24. If $u \in \mathbb{N}$ is given then I in (3.105) is a common isomorphic map for all $\bar{A}^s_{pq}(\Omega)$ with (3.102), (3.106) onto respective sequence spaces. This gives the possibility to shift mapping problems between these function spaces to corresponding assertions between sequence spaces where they often can be treated more easily. We refer to [DNS06a], [DNS06b], [DNS07] and the literature listed there. In numerical analysis preference is given (so far) to bounded Lipschitz domains which are covered by the above considerations.

Corollary 3.25. *Let Ω be either a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition 3.4* (iii) *or the classical snowflake domain from Proposition 3.8* (iii) *and Figure 3.5, p. 76, in $\mathbb{R}^2$. Then Ω is uniformly E-porous according to Definition 3.16* (ii) *and Theorem 3.23 can be applied.*

Proof. By Proposition 3.8 the domain Ω is E-thick in both cases. If Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ then $\Gamma = \partial\Omega$ is an $(n-1)$-set. The boundary Γ of the above snowflake domain is the so-called snowflake curve (Koch curve). It is an d-set with $d = \frac{\log 4}{\log 3}$ (Hausdorff dimension). We refer, for example, to [Mat95], pp. 66–68. Some relevant calculations may also be found in [T06], pp. 359–60. Then it follows from Proposition 3.18 (iii) that Ω is uniformly E-porous. In particular, Theorem 3.23 can be applied. □

Remark 3.26. There is little doubt that one can find for any d with $n-1 \leq d < n$ a bounded E-thick domain Ω in $\mathbb{R}^n$, $n \geq 2$, such that $\Gamma = \partial\Omega$ is an d-set. Then it follows from Proposition 3.18 (iii) that Ω is uniformly E-porous and one can apply Theorem 3.23. We refer to [T97], Section 16.4, p. 123, based on [T97], Theorem 16.2, Remark 16.3, pp. 120–22, where we constructed bounded star-like domains Ω such that $\Gamma = \partial\Omega$ is an d-set with $n-1 < d < n$. However it was noticed by A. Carvalho that some arguments given there are not correct. He gave a modified proof in [Car06]. The construction produces (presumably) also an E-thick domain. In Theorem 5.43 we complement the above assertions in case of so-called bounded cellular domains, covering some spaces so far excluded in Theorem 3.23.

3.3 Homogeneity and refined localisation, revisited

3.3.1 Introduction

We return to Section 2.2. For the spaces $\overline{A}^s_{pq}(U_\lambda)$ from Definition 2.9 we claimed the homogeneity (2.43). According to Remark 2.12 we know this property so far for the spaces $\overline{F}^s_{pq}(U_\lambda)$ with (2.45), hence the first line in (2.40), but not for the other cases. It will be our first aim to seal this gap now. This will be done in Section 3.3.2. It justifies afterwards the pointwise multiplier Theorem 2.13 for all cases and the assertions about the refined localisation spaces from Section 2.2.3. This will not be repeated. In Proposition 3.10 we proved that

$$F^{s,\mathrm{rloc}}_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega), \quad 0 < p,q \leq \infty, \ s > \sigma_{pq}, \tag{3.108}$$

($q = \infty$ if $p = \infty$) extending (2.63) from bounded C^∞ domains, [T01], and bounded Lipschitz domains, [T06], to E-thick domains. It is the second aim of the present Section 3.3 to prove

$$F^{s,\mathrm{rloc}}_{pq}(\Omega) = F^s_{pq}(\Omega), \quad 0 < p,q \leq \infty, \ s < 0, \tag{3.109}$$

($q = \infty$ if $p = \infty$) for the other spaces in Definition 2.14 if Ω is E-thick, complemented by a corresponding assertion if $s = 0$. This will be done in Section 3.3.3. Both (3.108) and (3.109) (complemented by a corresponding assertion for $s = 0$) cover also (2.42) what justifies finally the notation used in (2.40).

3.3.2 Homogeneity: Proof of Theorem 2.11

It does not matter whether U_λ in (2.39) are balls or cubes. To fix the imagination let us assume that

$$U_\lambda = \{x \in \mathbb{R}^n : \ |x| < \lambda\}, \quad 0 < \lambda \le 1, \tag{3.110}$$

are balls. Let $U = U_1$. Let $\bar{A}^s_{pq}(U_\lambda)$ with $A = B$ or $A = F$ be one of the spaces according to Definition 2.9. Then we have to prove that

$$\| f(\lambda \cdot) \,|\bar{A}^s_{pq}(U)\| \sim \lambda^{s-\frac{n}{p}} \, \| f \,|\bar{A}^s_{pq}(U_\lambda)\|, \quad 0 < \lambda \le 1, \tag{3.111}$$

where the equivalence constants are independent of λ and of $f \in \bar{A}^s_{pq}(U_\lambda)$. As mentioned in Remark 2.12 this assertion is known for

$$F^{s,\mathrm{rloc}}_{pq}(U_\lambda) = \bar{F}^s_{pq}(U_\lambda) = \tilde{F}^s_{pq}(U_\lambda) \quad \text{with } s > \sigma_{pq},$$

hence the first line in (2.40). This crucial observation has been used several times also after Section 2.2. It is one of the cornerstones of the theory developed in this book. Now we prove the remaining assertions which we did not use after Section 2.2, especially not to obtain the wavelet expansions in Theorems 3.13, 3.23 (Corollary 3.25) on which we now rely.

Step 1. We begin with a preparation. Let $A^s_{pq}(\Omega)$ be one of the spaces according to Definition 2.1 (i) in a domain Ω in $\mathbb{R}^n$. Let $0 < c_1 < c_2 < \infty$, typically $c_1 = 1/2$ and $c_2 = 2$, and let $c_1 < c < c_2$. Let

$$c\,\Omega = \{y \in \mathbb{R}^n : \ y = cx, \ x \in \Omega\}. \tag{3.112}$$

Then

$$\| f(c^{-1} \cdot) \,|A^s_{pq}(c\Omega)\| \sim \| f \,|A^s_{pq}(\Omega)\|, \quad f \in A^s_{pq}(\Omega), \tag{3.113}$$

where the equivalence constants depend on c_1, c_2, but not on c and Ω. This is known (and can be checked easily) if $\Omega = \mathbb{R}^n$. One obtains this assertion for arbitrary domains Ω by restriction according to Definition 2.1 (i). Similarly for (3.113) with $\tilde{A}^s_{pq}(\Omega)$ according to Definition 2.1 (ii) in place of $A^s_{pq}(\Omega)$.

Step 2. By Step 1 it is sufficient to deal with $\lambda = 2^{-k}$ where $k \in \mathbb{N}_0$. Let for brevity $U_{2^{-k}} = U^k$ in (3.111) with $U = U^0$ (the unit ball). With $\Omega = U_\lambda$ in Definition 3.11 one obtains the spaces in Definition 2.9. Hence we have for all these spaces wavelet representations which we constructed explicitly in the proofs of Proposition 3.10 ($s > \sigma_{pq}$), Theorem 3.13 ($s \neq 0$) and Proposition 3.21 ($s = 0$). In all cases we obtained the same type of representation, say, for example (3.49) with (3.50), based on the orthonormal u-wavelet basis $\{\Phi^j_r\}$ in (3.47). This gives the possibility to compare such expansions in two domains which are related to each other by a dilation of type (3.112). This applies in particular to the unit ball U as a reference domain and the dilated balls U^k. In this dyadic situation the functions $\{\Phi^j_r\}$ in U are transformed into the corresponding functions in U^k. But this applies not only to the functions $\{\Phi^j_r\}$ but also to the complements of the kernels k^j_r in (3.37) by (3.39) and of the atoms a^j_r in (3.55) by (3.56) if moment conditions are needed. Afterwards one has only $\mathbb{R}^n$-estimates, both for local means and for atoms based exclusively on Theorems 1.7, 1.15. In other words the equivalence constants for the isomorphic map I in (3.51) can be chosen in all cases covered by Theorem 3.13, hence $s \neq 0$, independently of $k \in \mathbb{N}$. This applies afterwards also to $s = 0$ by the arguments in the proof of Proposition 3.21. In other words it remains to control what happens with the coefficients in the related expansions if one applies the dilation

$$D_k : x \mapsto 2^{-k}x, \quad U^k = D_k U, \quad k \in \mathbb{N}. \tag{3.114}$$

We deal with the case $s \neq 0$ covered by Theorem 3.13. One can incorporate $s = 0$ afterwards by the arguments in the proof of Proposition 3.21 what will not be done in detail. Let $f \in \bar{A}^s_{pq}(U^k)$ covered by Theorem 3.13 where again $A = B$ or $A = F$. Similarly $a^s_{pq}(\mathbb{Z}_k)$ with $a = b$ or $a = f$ and $\mathbb{Z}_k = \mathbb{Z}_{U^k}$, where $\mathbb{Z}_0 = \mathbb{Z}_U$ refers to the unit ball U. Then

$$\| f \,|\bar{A}^s_{pq}(U^k)\| \sim \|\lambda(f)_k \,|a^s_{pq}(\mathbb{Z}_k)\| \tag{3.115}$$

with $\lambda(f)_k = \{\lambda^j_r(f)_k\}$, where

$$\lambda^j_r(f)_k = 2^{jn/2} \int_{U^k} f(x)\, \Phi^j_r(x)_k \,\mathrm{d}x, \quad (j,r) \in \mathbb{Z}_k, \tag{3.116}$$

(appropriately interpreted if $s < 0$) where $\Phi^j_r(\cdot)_k$ refers to the system (3.47) with $\Omega = U^k$. Let $\Phi^j_r(\cdot) = \Phi^j_r(\cdot)_0$ and $\lambda^j_r(f) = \lambda^j_r(f)_0$. On the other hand we can expand $f(2^{-k}\cdot) \in \bar{A}^s_{pq}(U)$ according to, say, (3.49) by

$$f\big(2^{-k}x\big) = \sum_{(j,r)\in\mathbb{Z}_0} \lambda^j_r\, \big(f(2^{-k}\cdot)\big)\, 2^{-jn/2}\, \Phi^j_r(x) \tag{3.117}$$

with (3.50), hence

$$\lambda^j_r\big(f(2^{-k}\cdot)\big) = 2^{jn/2} \int_U f(2^{-k}x)\, \Phi^j_r(x)\,\mathrm{d}x, \quad (j,r) \in \mathbb{Z}_0. \tag{3.118}$$

By the constructions resulting in Theorem 2.33, based on Definitions 2.4, 2.31, the wavelet bases for $L_2(U)$ and $L_2(U^k)$ are related by

$$\Phi^j_r(2^k x) = 2^{-kn/2}\, \Phi^{j+k}_r(x)_k, \quad x \in U_k,\ (j,r) \in \mathbb{Z}_0, \tag{3.119}$$

with the left-hand side considered as an orthonormal basis in $L_2(U)$ and the right-hand side considered as an orthonormal basis in $L_2(U^k)$. To stick at the same r it is suitable to accept an index-shifting in $\mathbb{Z}_k$ for j from $\mathbb{N}_0$ to $k+\mathbb{N}_0$. Then one has by (3.118) that

$$\lambda^j_r\big(f(2^{-k}\cdot)\big) = 2^{(j+k)n/2}\int_{U^k} f(x)\,\Phi^{j+k}_r(x)_k\,\mathrm{d}x = \lambda^{j+k}_r(f)_k. \tag{3.120}$$

Using (3.119), (3.120) one obtains by (3.117) that

$$\begin{aligned} f(x) &= \sum_{(j,r)\in\mathbb{Z}_0} \lambda^{j+k}_r(f)_k\,2^{-jn/2}\,\Phi^j_r(2^kx) \\ &= \sum_{(j,r)\in\mathbb{Z}_k} \lambda^j_r(f)_k\,2^{-jn/2}\,\Phi^j_r(x)_k, \quad x\in U^k, \end{aligned} \tag{3.121}$$

where we replaced $j+k$ by j in the last line and used the above index-shifting in $\mathbb{Z}_k$. This is the desired representation both for the B-spaces and the F-spaces.

Step 3. After these preparations one can now prove (3.111) with $\lambda = 2^{-k}$ and $U_\lambda = U^k$ as follows. By Theorem 3.13 (ii) and Definition 2.6 we have

$$\begin{aligned} \|f(2^{-k}\cdot)\,|\bar{B}^s_{pq}(U)\| &\sim \big\|\lambda\big(f(2^{-k}\cdot)\big)\,|b^s_{pq}(\mathbb{Z}_0)\big\| \\ &= \bigg(\sum_j 2^{j(s-\frac{n}{p})q}\Big(\sum_r \big|\lambda^j_r\,\big(f(2^{-k}\cdot)\big)\big|^p\Big)^{q/p}\bigg)^{1/q}. \end{aligned} \tag{3.122}$$

Inserting (3.120) one obtains by the above-mentioned index-shifting

$$\begin{aligned} \|f(2^{-k}\cdot)\,|\bar{B}^s_{pq}(U)\| &\sim 2^{-k(s-\frac{n}{p})}\,\|\lambda(f)_k\,|b^s_{pq}(\mathbb{Z}_k)\| \\ &\sim 2^{-k(s-\frac{n}{p})}\,\|f\,|\bar{B}^s_{pq}(U^k)\|. \end{aligned} \tag{3.123}$$

This proves (3.111) for the B-spaces. Similarly it follows for the F-spaces from Theorem 3.13 (i) and Definition 2.6 that

$$\begin{aligned} &\|f(2^{-k}\cdot)\,|\bar{F}^s_{pq}(U)\| \sim \big\|\lambda\big(f(2^{-k}\cdot)\big)|f^s_{pq}(\mathbb{Z}_0)\big\| \\ &= \bigg\|\Big(\sum_{(j,r)\in\mathbb{Z}_0} 2^{jsq}\,\big|\lambda^j_r\big(f(2^{-k}\cdot)\big)\,\chi_{jr}(\cdot)\big|^q\Big)^{1/q}|L_p(U)\bigg\| \\ &\sim 2^{-sk}\bigg\|\Big(\sum_{(j,r)\in\mathbb{Z}_0} 2^{(j+k)sq}\,\big|\lambda^{j+k}_r(f)_k\,\chi_{j+k,r}(2^{-k}\cdot)\big|^q\Big)^{1/q}|L_p(U)\bigg\| \\ &\sim 2^{-k(s-\frac{n}{p})}\bigg\|\Big(\sum_{(j,r)\in\mathbb{Z}_k} 2^{jsq}\,\big|\lambda^j_r(f)_k\,\chi_{j,r}(\cdot)\big|^q\Big)^{1/q}|L_p(U^k)\bigg\| \\ &\sim 2^{-k(s-\frac{n}{p})}\,\big\|f\,|\bar{F}^s_{pq}(U^k)\big\|. \end{aligned} \tag{3.124}$$

This proves (3.111) for the F-spaces.

3.3.3 Wavelet bases in $F^{s,\mathrm{rloc}}_{pq}(\Omega)$, revisited

After the complete proof of Theorem 2.11 we know now that also all the other assertions in Section 2.2 are valid. This applies in particular to the pointwise multiplier Theorem 2.13 and the properties of the refined localisation spaces according to Definition 2.14 and Theorem 2.16. For the spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ with $s > \sigma_{pq}$ we have the wavelet representation in Theorem 2.38. It is the first aim of this Section 3.3.3 to extend this assertion to other refined localisation spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ with $s \le 0$. Secondly we obtain as a corollary (3.109) under the assumption that Ω is E-thick. This can be rephrased as the *refined localisation property* of the related spaces $F^s_{pq}(\Omega)$.

We use the same notation and references as in Theorem 2.38.

Theorem 3.27. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ be the spaces as introduced in Definition 2.14. Let $\{\Phi^j_r\}$ be an orthonormal u-wavelet basis in $L_2(\Omega)$ according to Theorem 2.33 and Definitions 2.31, 2.4 with*

$$u > \max(s, \sigma_{pq} - s). \tag{3.125}$$

Then $f \in D'(\Omega)$ is an element of $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r \, 2^{-jn/2} \, \Phi^j_r, \quad \lambda \in f^s_{pq}(\mathbb{Z}_\Omega). \tag{3.126}$$

Furthermore, if $f \in F^{s,\mathrm{rloc}}_{pq}(\Omega)$ then the representation (3.126) is unique with $\lambda = \lambda(f)$,

$$\lambda^j_r(f) = 2^{jn/2} \left(f, \Phi^j_r\right) = 2^{jn/2} \int_\Omega f(x) \, \Phi^j_r(x) \, \mathrm{d}x \tag{3.127}$$

(appropriately interpreted) and

$$I : f \mapsto \lambda(f) = \left\{2^{jn/2}\left(f, \Phi^j_r\right)\right\} \tag{3.128}$$

is an isomorphic map of $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ onto $f^s_{pq}(\mathbb{Z}_\Omega)$,

$$\|f \,|F^{s,\mathrm{rloc}}_{pq}(\Omega)\| \sim \|\lambda(f) \,|f^s_{pq}(\mathbb{Z}_\Omega)\| \tag{3.129}$$

(equivalent quasi-norms). If $p < \infty$, $q < \infty$ then $\{\Phi^j_r\}$ is an unconditional basis in $F^{s,\mathrm{rloc}}_{pq}(\Omega)$.

Proof. Step 1. If $s > \sigma_{pq}$ (and hence $u > s$) then the above assertion coincides essentially with Theorem 2.38. Hence one has to prove the theorem for the remaining spaces with $s \le 0$. As far as formulations are concerned one may also consult Theorems 3.13 (i) and 3.23 (i).

Step 2. Let $s \le 0$ and let f be given by (3.126). We decompose f with respect to the Whitney cubes Q^1_{lt} in (2.22),

$$f = \sum_{l=0}^{\infty} \sum_{t} f_{lt} \quad \text{with } f_{lt} = \sum_{j,r} \lambda^j_r(l,t) \, 2^{-jn/2} \, \Phi^j_r, \tag{3.130}$$

where each f_{lt} is a partial sum of (3.126) collecting terms subordinated to Q^1_{lt} by the construction in (2.22), (2.23) indicated by $\lambda^j_r(l,t)$ which are either λ^j_r or zero such that there is an one-to-one relation between

$$\{\lambda^j_m\} \quad \text{and} \quad \bigcup_{l,t} \{\lambda^j_r(l,t)\}. \tag{3.131}$$

By the homogeneity (2.43) with $\bar{A}^s_{pq} = F^s_{pq}$ (now justified) and Theorem 3.23 (i) with $\bar{F}^s_{pq} = F^s_{pq}$ it follows that

$$\| f_{lt} \, | F^s_{pq}(Q^1_{lt} \| \sim \| \lambda(l,t) \, | f^s_{pq}(\mathbb{Z}_\Omega) \|, \tag{3.132}$$

where $\lambda(l,t)$ collects the related terms $\lambda^j_r(l,t)$ with equivalence constants which are independent of l and t. One has

$$\sum_{l,t} \| f_{l,t} \, | F^s_{pq}(Q^1_{lt} \|^p \sim \| \lambda \, | f^s_{pq}(\mathbb{Z}_\Omega) \|^p \tag{3.133}$$

(modification if $p = q = \infty$). Let $\{\varrho_{lt}\}$ be a resolution of unity as used in Definition 2.14. Then

$$\varrho_{lt} f = \varrho_{lt} \, (f_{lt} + +) \tag{3.134}$$

where $++$ indicates some neighbouring terms. By the pointwise multiplier Theorem 2.13 applied to $g = \varrho_{lt}$ one obtains by (2.59) and (3.133), (3.134) that $f \in F^{s,\mathrm{rloc}}_{pq}(\Omega)$ and

$$\| f \, | F^{s,\mathrm{rloc}}_{pq}(\Omega) \| \le c \, \| \lambda \, | f^s_{pq}(\mathbb{Z}_\Omega) \|. \tag{3.135}$$

By the orthogonality of $\{\Phi^j_r\}$ one has in the same way as before $\lambda^j_r = \lambda^j_r(f)$ with (3.127).

Step 3. Conversely let $f \in F^{s,\mathrm{rloc}}_{pq}(\Omega)$. Then we have $\varrho_{lt} f \in F^s_{pq}(Q^1_{lt})$ and one obtains by Theorem 3.23 (i) with $\bar{F}^s_{pq} = F^s_{pq}$,

$$\| \lambda(\varrho_{lt} f) \, | f^s_{pq}(\mathbb{Z}_\Omega) \| \sim \| \varrho_{lt} f \, | F^s_{pq}(Q^1_{lt}) \|. \tag{3.136}$$

Again it follows from the homogeneity (2.43) that the equivalence constants in (3.136) are independent of l and t (explicit calculations may be found in Step 3 of the proof in Section 3.3.2). But then it follows from (2.55), (2.59) that $\lambda(f) \in f^s_{pq}(\mathbb{Z}_\Omega)$ and

$$\| \lambda(f) \, | f^s_{pq}(\mathbb{Z}_\Omega) \| \le c \, \| f \, | F^{s,\mathrm{rloc}}_{pq}(\Omega) \|. \tag{3.137}$$

The representability of $f \in F^{s,\mathrm{rloc}}_{pq}(\Omega)$ by (3.126) (with $\lambda = \lambda(f)$) follows by the above arguments from the representability of $\varrho_{lt} f \in F^s_{pq}(Q^1_{lt})$. □

As before we put $F^s_{\infty\infty} = B^s_{\infty\infty}$.

Theorem 3.28. (i) *Let* Ω *be an* E*-thick domain in* $\mathbb{R}^n$ *according to Definition* 3.1 (ii). *Let*

$$\left.\begin{array}{l} \text{either } 0 < p \le \infty,\ 0 < q \le \infty,\ s > \sigma_{pq}, \\ \text{or } 0 < p \le \infty,\ 0 < q \le \infty,\ s < 0 \end{array}\right\} \tag{3.138}$$

($q = \infty$ *if* $p = \infty$). *Let* $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ *and* $\bar{F}^s_{pq}(\Omega)$ *be the spaces according to Definitions* 2.14 *and* 3.11. *Then*

$$F^{s,\mathrm{rloc}}_{pq}(\Omega) = \bar{F}^s_{pq}(\Omega). \tag{3.139}$$

(ii) *Let* Ω *be an uniformly porous domain in* $\mathbb{R}^n$ *according to Definition* 3.16 (ii). *Then*

$$F^{0,\mathrm{rloc}}_{pq}(\Omega) = F^0_{pq}(\Omega), \quad 1 < p < \infty,\ 1 \le q < \infty, \tag{3.140}$$

where $F^{0,\mathrm{rloc}}_{pq}(\Omega)$ *has the same meaning as in Definition* 2.14.

Proof. If $s > \sigma_{pq}$ then (3.139) is covered by Proposition 3.10 as a consequence of the fact that the two spaces have the same wavelet representations. But this applies also to all other cases, comparing Theorem 3.13, Proposition 3.21 with Theorem 3.27. □

Remark 3.29. In particular one obtains (3.109) as a special case, complementing (3.108).

3.3.4 Duality

Recall the well-known duality assertion

$$B^s_{pq}(\mathbb{R}^n)' = B^{-s}_{p'q'}(\mathbb{R}^n), \quad s \in \mathbb{R}, \tag{3.141}$$

with

$$1 \le p, q < \infty, \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1, \tag{3.142}$$

where $B^s_{pq}(\mathbb{R}^n)$ are the spaces according to Definition 1.1. Here (3.141) must be interpreted within the dual pairing $(S(\mathbb{R}^n), S'(\mathbb{R}^n))$ or the dual pairing $(D(\mathbb{R}^n), D'(\mathbb{R}^n))$. This requires that $S(\mathbb{R}^n)$ and $D(\mathbb{R}^n)$ are dense in $B^s_{pq}(\mathbb{R}^n)$, what is the case (since we excluded $p = \infty$ and/or $q = \infty$ in (3.142)). This is a classical assertion which may be found in [T78], Section 2.6.1, pp. 198–99, and [T83], Theorem 2.11.2, p. 178. It can be extended to spaces with $p < 1$ and also to the spaces $F^s_{pq}(\mathbb{R}^n)$. We refer again to [T83], Section 2.11. But this is not of interest in the present context. We ask for counterparts of (3.141) with domains Ω in place of $\mathbb{R}^n$. There are assertions of this type for the spaces $\tilde{B}^s_{pq}(\bar{\Omega})$ according to Definition 2.1 with respect to arbitrary domains Ω in [T78], Section 4.8.1, p. 332, with the expected outcome. But this was a little bit too bold at least as far as the interpretation of $\tilde{B}^s_{pq}(\bar{\Omega})$ as a subspace of $D'(\Omega)$ is concerned. In some sense we seize the opportunity and return to this point (after more than thirty years) using now the above wavelet characterisations.

Theorem 3.30. *Let $\overline{B}^s_{pq}(\Omega)$ be the same spaces as in* (3.46) *where Ω is an E-thick domain in $\mathbb{R}^n$ according to Definition* 3.1 (ii) *with*

$$0 \neq s \in \mathbb{R}, \quad 1 \le p, q < \infty, \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1. \tag{3.143}$$

Then $D(\Omega)$ is dense in $\overline{B}^s_{pq}(\Omega)$ and

$$\overline{B}^s_{pq}(\Omega)' = \overline{B}^{-s}_{p'q'}(\Omega) \tag{3.144}$$

are the dual spaces interpreted within the dual pairing $(D(\Omega), D'(\Omega))$.

Proof. Step 1. Let s, p, q be as in (3.143). It follows from Theorem 3.13 (ii) that I in (3.51) is an isomorphic map onto $b^s_{pq}(\mathbb{Z}_\Omega)$ normed by (2.37). Then one obtains by (3.53) that finite linear combinations of Φ^j_r are dense in $\overline{B}^s_{pq}(\Omega)$. Since $\Phi^j_r \in C^u(\mathbb{R}^n)$ has compact support in Ω it can be approximated within Ω by $D(\Omega)$-functions (Sobolev's mollification). Hence $D(\Omega)$ is dense in $\overline{B}^s_{pq}(\Omega)$.

Step 2. Since $D(\Omega)$ is dense in $\overline{B}^s_{pq}(\Omega)$ with (3.143) it makes sense to interpret its dual within the dual pairing $(D(\Omega), D'(\Omega))$. We complement $b^s_{pq}(\mathbb{Z}_\Omega)$ in Definition 2.6 by $\ell_q(\ell_p^{N_j})$ consisting of all sequences

$$\mu = \{\mu^j_r \in \mathbb{C} : \; j \in \mathbb{N}_0; \; r = 1, \ldots, N_j\}, \quad N_j \in \overline{\mathbb{N}}, \tag{3.145}$$

with

$$\|\mu \,|\ell_q(\ell_p^{N_j})\| = \Big(\sum_{j=0}^{\infty} \Big(\sum_{r=1}^{N_j} |\mu^j_r|^p \Big)^{q/p} \Big)^{1/q} < \infty. \tag{3.146}$$

If $1 \le p, q < \infty$ then one has

$$\ell_q(\ell_p^{N_j})' = \ell_{q'}(\ell_{p'}^{N_j}) \tag{3.147}$$

for the dual spaces interpreted as usual as the dual pairing

$$\langle \mu, \mu' \rangle = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \mu^j_r \, \mu'^j_r, \quad \mu \in \ell_q(\ell_p^{N_j}), \; \mu' \in \ell_{q'}(\ell_{p'}^{N_j}). \tag{3.148}$$

To adapt I in (3.51) to $\ell_q(\ell_p^{N_j})$ we put

$$J : \; f \mapsto \mu(f) = \{\mu^j_r(f)\}, \quad N_j \in \overline{\mathbb{N}} \tag{3.149}$$

with

$$\mu^j_r(f) = 2^{j(s - \frac{n}{p} + \frac{n}{2})} \int_\Omega f(x)\, \Phi^j_r(x)\, \mathrm{d}x \tag{3.150}$$

(usually interpreted if $s < 0$). Then

$$J\, \overline{B}^s_{pq}(\Omega) = \ell_q(\ell_p^{N_j}). \tag{3.151}$$

For the dual operator J' one has the isomorphic map

$$J' : \ell_{q'}(\ell_{p'}^{N_j}) = \ell_q(\ell_p^{N_j})' \quad \text{onto} \quad \overline{B}^s_{pq}(\Omega)', \tag{3.152}$$

interpreting the dual of $\overline{B}^s_{pq}(\Omega)$ as a subspace of $D'(\Omega)$. In particular one obtains with $f \in \overline{B}^s_{pq}(\Omega)$ and $\mu' \in \ell_{q'}(\ell_{p'}^{N_j})$ that

$$\begin{aligned} \langle Jf, \mu' \rangle &= \sum_{j,r} \mu'^j_r \, 2^{j(s-\frac{n}{p}+\frac{n}{2})} \int_\Omega f(x)\, \Phi^j_r(x)\, \mathrm{d}x \\ &= \int_\Omega f(x) \sum_{j,r} \mu'^j_r \, 2^{j(s-\frac{n}{p}+\frac{n}{2})}\, \Phi^j_r(x)\, \mathrm{d}x \\ &= (f, J'\mu'). \end{aligned} \tag{3.153}$$

Hence

$$J'\mu' = \sum_{j,r} \lambda'^j_r \, 2^{-jn/2}\, \Phi^j_r \tag{3.154}$$

with

$$\lambda'^j_r = 2^{j(s-\frac{n}{p}+n)}\, \mu'^j_r = 2^{-j(-s-\frac{n}{p'})}\, \mu'^j_r. \tag{3.155}$$

One obtains

$$\|\lambda' \,|b^{-s}_{p'q'}(\mathbb{Z}_\Omega)\| = \|\mu' \,|\ell_{q'}(\ell_{p'}^{N_j})\|. \tag{3.156}$$

Then (3.144) follows from (3.154), (3.156) and Theorem 3.13. □

Remark 3.31. We excluded $s = 0$ in Theorem 3.30 because one needs in this case that the underlying domain Ω is not only E-thick but even uniformly E-porous what is a stronger assumption according to Definition 3.16 and Proposition 3.18 (i).

Corollary 3.32. *Let $\overline{B}^0_{pq}(\Omega) = B^0_{pq}(\Omega)$ be the same spaces as in (3.46) where Ω is an uniformly E-porous domain in $\mathbb{R}^n$ according to Definition 3.16* (ii). *If*

$$1 < p < \infty, \quad 1 \le q < \infty, \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1, \tag{3.157}$$

then $D(\Omega)$ is dense in $B^0_{pq}(\Omega)$ and

$$B^0_{pq}(\Omega)' = B^0_{p'q'}(\Omega) \tag{3.158}$$

are the dual spaces interpreted within the dual pairing $(D(\Omega), D'(\Omega))$.

Proof. One can argue in the same way as in the proof of Theorem 3.30 using now Proposition 3.21 (ii) instead of Theorem 3.13 (ii). □

Corollary 3.33. *The assertions of Theorem 3.30 and Corollary 3.32 apply in particular to the spaces $\overline{B}^s_{pq}(\Omega)$ with (3.143) and $\overline{B}^0_{pq}(\Omega)$ with (3.157) if Ω is either an uniformly E-porous domain according to Definition 3.16 or a bounded Lipschitz domain according to Definition 3.4* (iii) *or the classical snowflake domain from Proposition 3.8* (iii) *and Figure 3.5, p. 76, in $\mathbb{R}^2$.*

Proof. By Proposition 3.18 E-porous domains are in particular E-thick. Hence one can apply both Theorem 3.30 and Corollary 3.32. The corresponding assertions for bounded Lipschitz domains and the classical snowflake domain follow now from Corollary 3.25. □

Remark 3.34. The proofs of Theorem 3.30 and Corollary 3.32 rely on the existence of a common wavelet basis in $\bar{B}^s_{pq}(\Omega)$ and its anticipated dual. One can try to apply this argument to other cases. A distinguished example is the special case

$$B^{s,\mathrm{rloc}}_{pp}(\Omega) = F^{s,\mathrm{rloc}}_{pp}(\Omega), \quad s \in \mathbb{R},\ 1 < p < \infty, \tag{3.159}$$

of the refined localisation spaces according to Definition 2.14. We formulate the outcome.

Corollary 3.35. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Then $D(\Omega)$ is dense in the refined localisation spaces $B^{s,\mathrm{rloc}}_{pp}(\Omega)$ according to Definition 2.14 and (3.159). Furthermore,*

$$B^{s,\mathrm{rloc}}_{pp}(\Omega)' = B^{-s,\mathrm{rloc}}_{p'p'}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{3.160}$$

are the dual spaces interpreted within the dual pairing $(D(\Omega), D'(\Omega))$.

Proof. Using Theorem 3.27 one can apply the arguments from the proof of Theorem 3.30 with

$$f^s_{pp}(\mathbb{Z}_\Omega) = b^s_{pp}(\mathbb{Z}_\Omega) \tag{3.161}$$

according to Remark 2.7. □

Chapter 4
The extension problem

4.1 Introduction and criterion

4.1.1 Introduction

We introduced in Definition 2.1 the spaces $A^s_{pq}(\Omega)$ on arbitrary domains (= open sets) in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ by restriction of $A^s_{pq}(\mathbb{R}^n)$ to Ω. In particular, $A^s_{pq}(\Omega)$ is considered as a subset of $D'(\Omega)$ and re,

$$\text{re}\, g = g|\Omega \colon\ A^s_{pq}(\mathbb{R}^n) \hookrightarrow A^s_{pq}(\Omega), \tag{4.1}$$

is the linear and bounded *restriction operator* according to (2.2). One of the most fundamental problem in the theory of function spaces is the question of whether there is a linear and bounded extension operator ext,

$$\text{ext} \colon\ A^s_{pq}(\Omega) \hookrightarrow A^s_{pq}(\mathbb{R}^n), \quad \text{ext}\, f|\Omega = f. \tag{4.2}$$

Using (4.1) this can also be written as

$$\text{re} \circ \text{ext} = \text{id} \quad (\text{identity in } A^s_{pq}(\Omega)). \tag{4.3}$$

A first discussion including relevant references may be found in Remark 3.3. It is mainly a question about the quality of the underlying domain Ω. The nowadays classical assertions about extensions may be found in [T92], Theorem 5.1.3, p. 239, where Ω is a bounded C^∞ domain. The final solution of the extension problem for bounded Lipschitz domains Ω in $\mathbb{R}^n$ goes back to [Ry99] where V. S. Rychkov constructed an universal extension operator which applies to all spaces $A^s_{pq}(\Omega)$. For a corresponding formulation one may also consult [T06], Section 1.11.5. As we indicated in Remark 3.3 beyond Lipschitz domains the extension problem is rather tricky. It may happen that the constructed operator, for example Ext^k_p in (3.13) is not an extension operator for other spaces. To ensure that there are extension operators is of interest for its own sake, but there are applications where one needs common extension operators at least in some (s,p,q)-regions. In contrast to the literature mentioned above and in Remark 3.3 we deal with the extension problem in the framework of wavelet bases.

4.1.2 A criterion

Let Ω be a (non-empty) arbitrary domain (= open set) in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and let $\Omega^c = \mathbb{R}^n \setminus \Omega$ be its (non-empty) closed complement in $\mathbb{R}^n$. Let s, p, q be as in (2.3) with $p < \infty$ for the F-spaces. Then

$$\tilde{A}^s_{pq}(\Omega^c) = \{f \in A^s_{pq}(\mathbb{R}^n) : \operatorname{supp} f \subset \Omega^c\} \tag{4.4}$$

is a closed subspace of $A^s_{pq}(\mathbb{R}^n)$. This extends (2.6) to arbitrary closed (non-empty) sets $\Gamma = \Omega^c$ in $\mathbb{R}^n$ with $\Gamma \neq \mathbb{R}^n$. At the beginning of Section 3.2.3 we recalled that a closed subspace of a quasi-Banach space is called *complemented* if there is a *projection* P with (3.68), (3.69). There we explained also that $\hookrightarrow$ means linear and continuous embedding.

Theorem 4.1. *Let Ω be an arbitrary* (*non-empty*) *domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and let $A^s_{pq}(\Omega)$ be a space according to Definition 2.1. Then there exists an extension operator* ext *with* (4.2) *if, and only if, $\tilde{A}^s_{pq}(\Omega^c)$ is a complemented subspace of $A^s_{pq}(\mathbb{R}^n)$.*

Proof. Step 1. Let ext be an extension operator for $A^s_{pq}(\Omega)$ according to (4.2), (4.3). Then

$$P = \operatorname{ext} \circ \operatorname{re} \colon A^s_{pq}(\mathbb{R}^n) \hookrightarrow A^s_{pq}(\mathbb{R}^n) \tag{4.5}$$

is a linear and bounded map. By (4.3) one has

$$P^2 = \operatorname{ext} \circ \operatorname{re} \circ \operatorname{ext} \circ \operatorname{re} = P. \tag{4.6}$$

Hence P is a projection of $A^s_{pq}(\mathbb{R}^n)$ onto $PA^s_{pq}(\mathbb{R}^n)$. By $P \circ \operatorname{ext} = \operatorname{ext}$ it follows that ext maps $A^s_{pq}(\Omega)$ into $PA^s_{pq}(\mathbb{R}^n)$. If $f = PA^s_{pq}(\mathbb{R}^n)$ then $f = \operatorname{ext}[\operatorname{re} f]$. Hence ext maps $A^s_{pq}(\Omega)$ onto $PA^s_{pq}(\mathbb{R}^n)$. From

$$\|f \,|A^s_{pq}(\Omega)\| \sim \|\operatorname{ext} f \,|A^s_{pq}(\mathbb{R}^n)\| \tag{4.7}$$

if follows that

$$\operatorname{ext} \colon A^s_{pq}(\Omega) \Longleftrightarrow PA^s_{pq}(\mathbb{R}^n) \tag{4.8}$$

is an isomorphic map. Since P is a projection one obtains that $Q = \mathrm{id} - P$ is also a projection. Furthermore, since

$$Qf|\Omega = f|\Omega - Pf|\Omega = 0, \quad f \in A^s_{pq}(\mathbb{R}^n), \tag{4.9}$$

one obtains that $\operatorname{supp} Qf \subset \Omega^c$. Conversely if $f \in A^s_{pq}(\mathbb{R}^n)$ with $\operatorname{supp} f \subset \Omega^c$ then $Pf = 0$ and hence $f = Qf$. In other words, $QA^s_{pq}(\mathbb{R}^n) = \tilde{A}^s_{pq}(\Omega^c)$. Hence $\tilde{A}^s_{pq}(\Omega^c)$ is a complemented subspace of $A^s_{pq}(\mathbb{R}^n)$.

Step 2. Let $\tilde{A}^s_{pq}(\Omega^c)$ be a complemented subspace of $A^s_{pq}(\mathbb{R}^n)$ and let Q be the corresponding projection. Let $P = \mathrm{id} - Q$ and

$$\operatorname{ext} f = Pg, \quad f \in A^s_{pq}(\Omega),\ g \in A^s_{pq}(\mathbb{R}^n),\ g|\Omega = f. \tag{4.10}$$

If $h \in \tilde{A}^s_{pq}(\Omega^c)$ then $Ph = h - Qh = 0$. Hence, $\operatorname{ext} f$ is independent of the chosen g with the indicated properties. In particular, ext is a linear operator, and

$$\|\operatorname{ext} f \,|A^s_{pq}(\mathbb{R}^n)\| \le c \inf_{g|\Omega = f} \|g \,|A^s_{pq}(\mathbb{R}^n)\| \sim \|f \,|A^s_{pq}(\Omega)\|. \tag{4.11}$$

Furthermore,

$$\operatorname{ext} f|\Omega = f|\Omega - Qf|\Omega = f|\Omega. \tag{4.12}$$

Hence ext is a (linear and bounded) extension operator. □

Remark 4.2. If Q is a common projection in, say, $B^s_{pq}(\mathbb{R}^n)$ with

$$(s, p, q) \in R_B \subset \mathbb{R} \times (0, \infty] \times (0, \infty], \tag{4.13}$$

then ext is a common extension operator for the corresponding spaces $B^s_{pq}(\Omega)$ in the same region R_B. Similarly for the F-spaces, or both B-spaces and F-spaces. This follows from the construction. Here *common* means that, say, ext is defined on the union of the spaces in questions,

$$\operatorname{dom}(\operatorname{ext}) = \bigcup_{(s,p,q)\in R_B} B^s_{pq}(\Omega) \tag{4.14}$$

such that its restriction to each admitted space $B^s_{pq}(\Omega)$ has the indicated property.

Remark 4.3. In particular one has always extension operators for the Hilbert spaces $H^s(\Omega) = B^s_{2,2}(\Omega)$ with $s \in \mathbb{R}$ in arbitrary domains Ω. (Recall that an infinite-dimensional Banach space is isomorphic to a Hilbert space if, and only if, every closed subspace is complemented, [AlK06], Theorem 12.4.4, p. 305). But this does not say very much. We are looking for common extension operators in some (s, p, q)-regions as explained in the previous remark. We discussed these questions in Section 4.1.1 and in Remark 3.3 where one finds also some references.

4.2 Main assertions

4.2.1 Positive smoothness

In (4.2) and Remark 4.2 we explained what is meant by an extension operator and a common extension operator. As in (3.44) we put

$$\sigma_p = n\Big(\frac{1}{p} - 1\Big)_+ \quad \text{and} \quad \sigma_{pq} = n\Big(\frac{1}{\min(p,q)} - 1\Big)_+, \quad 0 < p, q \le \infty. \tag{4.15}$$

The spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ have the same meaning as in Definition 2.1.

Theorem 4.4. *Let Ω be a (non-empty) I-thick domain in $\mathbb{R}^n$ according to Definition 3.1 (iii) with $\overline{\Omega} \neq \mathbb{R}^n$ and $|\partial\Omega| = 0$. Then for any $u > 0$ there is a common extension operator* ext_u*,*

$$\operatorname{ext}_u \colon B^s_{pq}(\Omega) \hookrightarrow B^s_{pq}(\mathbb{R}^n), \quad 0 < p \le \infty,\ 0 < q \le \infty,\ \sigma_p < s < u, \tag{4.16}$$

and

$$\operatorname{ext}_u \colon F^s_{pq}(\Omega) \hookrightarrow F^s_{pq}(\mathbb{R}^n), \quad 0 < p < \infty,\ 0 < q \le \infty,\ \sigma_{pq} < s < u. \tag{4.17}$$

Proof. One has by Proposition 3.6 (iv) that $\omega = \mathbb{R}^n \setminus \overline{\Omega}$ is E-thick. Furthermore since

$$\Omega^c = \mathbb{R}^n \setminus \Omega = \partial\Omega \cup \omega \quad \text{and} \quad |\partial\Omega| = 0 \tag{4.18}$$

one obtains by Proposition 3.15 for the above spaces, abbreviated by A^s_{pq},

$$\tilde{A}^s_{pq}(\Omega^c) = \tilde{A}^s_{pq}(\overline{\omega}) = \mathrm{id}\tilde{A}^s_{pq}(\omega). \tag{4.19}$$

Here we used (3.20), hence $\partial\omega \subset \partial\Omega$, $|\partial\omega| = 0$, and $s > \sigma_p$, hence $A^s_{pq}(\mathbb{R}^n) \subset L^{\mathrm{loc}}_1(\mathbb{R}^n)$. We deal with the F-spaces. Let for $u \in \mathbb{N}$,

$$\{\Phi^j_r : j \in \mathbb{N}_0;\ r = 1, \ldots, N_j\} \tag{4.20}$$

be an orthonormal u-wavelet bases now in $L_2(\omega)$ as in Theorem 3.13 and let

$$Qf = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \Lambda^j_r(f)\, 2^{-jn/2}\, \Phi^j_r, \quad f \in F^s_{pq}(\mathbb{R}^n), \tag{4.21}$$

with

$$\Lambda^j_r(f) = 2^{jn/2} \int_{\mathbb{R}^n} \tilde{k}^j_r(x)\, f(x)\, \mathrm{d}x \tag{4.22}$$

be the adapted counterpart of (3.74), (3.75). By (4.19) and the same arguments as in the proof of Proposition 3.15 it follows that Q is a projection in $F^s_{pq}(\mathbb{R}^n)$ and

$$QF^s_{pq}(\mathbb{R}^n) = \tilde{F}^s_{pq}(\Omega^c). \tag{4.23}$$

Now one obtains by Theorem 4.1 and Remark 4.2 that there is a common extension operator ext_u for all spaces in (4.17) which can be constructed explicitly by (4.10). These arguments apply also to the spaces in (4.16). □

Corollary 4.5. *Theorem 4.4 applies in particular to*

(i) *bounded Lipschitz domains Ω in $\mathbb{R}^n$, $n \geq 2$, according to Definition 3.4* (iii)*, or to intervals on $\mathbb{R}$,*

(ii) *(ε, δ)-domains Ω in $\mathbb{R}^n$ according to Definition 3.1* (i) *with $\overline{\Omega} \neq \mathbb{R}^n$,*

(iii) *snowflake domains Ω in $\mathbb{R}^2$, Figure 3.5, p. 76.*

Proof. By Propositions 3.6, 3.8 the above domains are I-thick with $\overline{\Omega} \neq \mathbb{R}^n$ and $|\partial\Omega| = 0$. This shows that one can apply Theorem 4.4. □

Remark 4.6. The discussions in Section 3.1.3 show that there exist rather bizarre (disconnected) I-thick domains Ω even if one assumes that $|\partial\Omega| = 0$. On the other hand, domains with inwards cusps as in Figure 3.3, p. 74, are I-thick and one can apply Theorem 4.4, whereas domains with outwards cusps as in Figure 3.4, p. 74, are not I-thick. Recall that the spaces $A^s_{pq}(\Omega)$ are defined by restriction. In rough domains (beyond Lipschitz domains) the question of extendability is different if one deal with spaces introduced intrinsically. This applies in particular to the Sobolev spaces $W^k_p(\Omega)$ according to Definition 2.53. We discussed this question in Remark 3.3 where one finds also some references.

4.2.2 Negative smoothness

The arguments in the proof of Theorem 4.4 cannot be applied to spaces $A^s_{pq}(\Omega)$ with $s < 0$. On the other hand we obtained in Theorem 3.13 wavelet bases in spaces $A^s_{pq}(\Omega)$ with $s < 0$ in a constructive way. This can be interpreted in terms of extension operators. As before (4.2) and Remark 4.2 say what is meant by common extension operators. The spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ have the same meaning as in Definition 2.1.

Theorem 4.7. *Let Ω be a (non-empty) E-thick domain in $\mathbb{R}^n$ according to Definition 3.1* (ii) *with $\Omega \neq \mathbb{R}^n$. Then for any ε with $0 < \varepsilon < 1$ there is a common extension operator* ext^ε,

$$\mathrm{ext}^\varepsilon : B^s_{pq}(\Omega) \hookrightarrow B^s_{pq}(\mathbb{R}^n), \quad -\varepsilon^{-1} < s < 0, \ \varepsilon < p \le \infty, \ 0 < q \le \infty, \tag{4.24}$$

and

$$\mathrm{ext}^\varepsilon : F^s_{pq}(\Omega) \hookrightarrow F^s_{pq}(\mathbb{R}^n), \quad -\varepsilon^{-1} < s < 0, \ \varepsilon < p < \infty, \ \varepsilon < q \le \infty. \tag{4.25}$$

Proof. We deal with the F-spaces and rely on the arguments in Step 2 of the proof of Theorem 3.13. Let

$$\big\{\Phi^j_r : j \in \mathbb{N}_0 : r = 1, \dots, N_j\big\}, \quad \sigma_{pq} - s < u \in \mathbb{N}, \tag{4.26}$$

be the same orthonormal u-wavelet basis in $L_2(\Omega)$ as in (3.47), (3.54). Let a^j_r be as in (3.55) (basic and interior wavelets Φ^j_r) and as in (3.56) (boundary wavelets Φ^j_r). For ε with $0 < \varepsilon < 1$ and given p, q we choose $\sigma_{pq} + \varepsilon^{-1} < u \in \mathbb{N}$. Then (4.26) applies to all s with $-\varepsilon^{-1} < s < 0$. By Theorem 3.13 any $f \in F^s_{pq}(\Omega)$ can be represented by (3.59), (3.60), hence

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r(f)\, 2^{-jn/2}\, \Phi^j_r, \quad \lambda^j_r(f) = 2^{jn/2}\, \big(f, \Phi^j_r\big). \tag{4.27}$$

Then ext^ε given by

$$\mathrm{ext}^\varepsilon f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} 2^{j(s-\frac{n}{p})}\, \lambda^j_r(f)\, a^j_r, \tag{4.28}$$

is a linear and bounded extension operator. This follows from $\mathrm{ext}^\varepsilon f|\Omega = f$ and (3.57), (3.62). Choosing $u > \varepsilon^{-1} + \sigma_{\varepsilon\varepsilon} = (n+1)\varepsilon^{-1} - n$ one has a common extension operator for all spaces in (4.25). Similarly for the B-spaces where the same extension operator ext^ε can also be applied to the spaces in (4.24). □

Corollary 4.8. *Theorem 4.7 applies in particular to*

(i) *bounded Lipschitz domains in $\mathbb{R}^n$, $n \ge 2$, according to Definition 3.4* (iii), *and intervals on $\mathbb{R}$,*

(ii) *snowflake domains Ω in $\mathbb{R}^2$, Figure 3.5, p. 76.*

Proof. This follows from Proposition 3.8. □

Remark 4.9. By Remark 3.7 E-thick domains Ω in $\mathbb{R}^n$ may be rather bizarre. It may even happen that $|\partial\Omega| > 0$. Nevertheless by Theorem 4.7 we have always linear and bounded extension operators. On the other hand the domains with inwards cusps as in Figure 3.3, p. 74, are not E-thick, but domains with outwards cusps as in Figure 3.4, p. 74, are E-thick. Otherwise we refer to our discussion in Remark 4.6 about intrinsically defined Sobolev spaces $W^k_p(\Omega)$.

4.2.3 Combined smoothness

If Ω is a thick domain in $\mathbb{R}^n$ according to Definition 3.1 (iv) (hence both E-thick and I-thick) with $\overline{\Omega} \neq \mathbb{R}^n$ and $|\partial\Omega| = 0$ then we can apply both Theorem 4.4 and Theorem 4.7 covering all spaces $\overline{A}^s_{pq}(\Omega)$ in Definition 3.11 with $s \neq 0$. However the extension operators from Theorem 4.4 for spaces with $s > 0$ and from Theorem 4.7 for spaces with $s < 0$ are different. There is little hope to find in this way common extension operators which apply simultaneously to spaces with positive and negative smoothness s and which include also spaces $A^0_{pq}(\Omega)$ of smoothness $s = 0$ covered by Definition 3.11. But there is a way to extend Theorem 4.7 to some spaces with $s \geq 0$ and to some domains with rough boundary. But one has to strengthen the assumptions about the underlying domain Ω.

First we recall some notation and assertions from Section 3.2.4. According to (3.79), (3.80) a closed set Γ in $\mathbb{R}^n$ is called an d-set, $0 < d < n$, if there is a Radon measure μ with

$$\mu(B(\gamma, r)) \sim r^d \quad \text{where } \gamma \in \Gamma = \operatorname{supp} \mu,\ 0 < r < 1. \tag{4.29}$$

The equivalence constants in (4.29) are independent of γ and r (in agreement with (3.4)). If Ω is an E-thick domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ such that its boundary $\Gamma = \partial\Omega$ is an d-set then one has by Proposition 3.18 (iii) that $n-1 \leq d < n$. In case of $n = 1$ one may incorporate now $d = 0$. Then (4.29) with $d = 0$ and the assumption that Ω is an E-thick domain on the real line $\mathbb{R}$ show that one may assume (without restriction of generality) that Ω is an interval.

Theorem 4.10. *Let Ω be a (non-empty) E-thick domain in $\mathbb{R}^n$ according to Definition 3.1 (ii) with $\Omega \neq \mathbb{R}^n$ such that its boundary $\Gamma = \partial\Omega$ is an d-set satisfying (4.29) with $n-1 \leq d < n$ (in case of $n = 1$ and $d = 0$ this means that Ω is an interval). Then for any ε with $0 < \varepsilon < 1$ there is a common extension operator ext^ε,*

$$\mathrm{ext}^\varepsilon : B^s_{pq}(\Omega) \hookrightarrow B^s_{pq}(\mathbb{R}^n), \quad \begin{cases} \varepsilon < p \leq \infty,\ 0 < q \leq \infty,\ -\varepsilon^{-1} < s < 0, \\ 1 < p < \infty,\ 0 < q \leq \infty,\ 0 \leq s < \frac{n-d}{p}, \end{cases} \tag{4.30}$$

and

$$\mathrm{ext}^\varepsilon : F^s_{pq}(\Omega) \hookrightarrow F^s_{pq}(\mathbb{R}^n), \quad \begin{cases} \varepsilon < p < \infty,\ \varepsilon < q \leq \infty,\ -\varepsilon^{-1} < s < 0, \\ 1 < p < \infty,\ 1 \leq q < \infty,\ 0 \leq s < \frac{n-d}{p}. \end{cases} \tag{4.31}$$

Proof. Let ext^ε be the same extension operator as in Theorem 4.7. Then (4.24), (4.25) coincides with the upper lines in (4.30), (4.31). Hence it remains to prove that ext^ε is also an extension operator for the corresponding lower lines. For $h(r) = r^d$ one has (3.82) with $\lambda = n - d$. By Proposition 3.18 (iii) and Step 2 of its proof it follows that Ω is uniformly E-porous and that Γ is porous. Then we can apply Proposition 3.19 specified to d-sets. As remarked in the proof of Proposition 3.19 one may choose $\delta = \lambda = n - d$ in case of d-sets. We deal with the F-spaces. Since Ω is uniformly E-porous one has by (3.87) and Theorem 3.23 the representation (4.27) = (3.103), (3.104), hence

$$\|\lambda(f)\,|f^s_{pq}(\mathbb{Z}_\Omega)\| \sim \|f\,|F^s_{pq}(\Omega)\|. \tag{4.32}$$

On the other hand, ext^ε in (4.28) is an atomic decomposition in $F^s_{pq}(\mathbb{R}^n)$ and one obtains as before with a reference to Theorem 1.7 that

$$\|\,\mathrm{ext}^\varepsilon f\,|F^s_{pq}(\mathbb{R}^n)\| \le c\,\|\lambda(f)\,|f^s_{pq}(\mathbb{Z}_\Omega)\|. \tag{4.33}$$

By (4.32) and $\mathrm{ext}^\varepsilon f|\Omega = f$ it follows that ext^ε is also an extension operator for the lower line in (4.31). Similarly for the B-spaces. □

Remark 4.11. If Ω is a bounded Lipschitz domain according to Definition 3.4 (iii) (or an interval in case of $n = 1$) then Ω is E-thick, $\Gamma = \partial\Omega$ is an $(n-1)$-set and one has $0 \le s < 1/p$ in the lower lines of (4.30), (4.31). In case of the snowflake domain in Figure 3.5, p. 76, one has $0 \le s < \frac{1}{p}\big(2 - \frac{\log 4}{\log 3}\big)$ in the lower lines in (4.30), (4.31). Otherwise we restricted ourselves in the above theorem to boundaries Γ which are d-sets according to (4.29). But it is quite obvious that one can generalise (4.29) by (3.79) with (3.82) for some $\lambda > 0$. Then one can extend Theorem 4.7 by a counterpart of Theorem 4.10 using Proposition 3.19 and Theorem 3.23.

Although quite obvious it seems to be desirable to give an explicit formulation covering both Theorem 4.4 and Theorem 4.10. In Remark 4.2 we said what is meant by a *common* extension operator. An extension operator is called *universal* if it applies to all spaces $A^s_{pq}(\Omega)$ with $s \in \mathbb{R}$, $0 < p,q \le \infty$ ($p < \infty$ for the F-spaces) in the understanding of Remark 4.2. The numbers σ_p and σ_{pq} have the same meaning as in (4.15).

Corollary 4.12. (i) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \ge 2$ according to Definition* 3.4 (iii) *or an interval on $\mathbb{R}$. Then there is a universal extension operator.*

(ii) *Let Ω be a (non-empty) thick domain in $\mathbb{R}^n$ according to Definition* 3.1 (iv) *with $\overline{\Omega} \ne \mathbb{R}^n$ such that its boundary $\Gamma = \partial\Omega$ is an d-set satisfying* (4.29) *with $n - 1 \le d < n$. Then there is for any $u > 0$ a common extension operator ext_u with*

$$\mathrm{ext}_u : B^s_{pq}(\Omega) \hookrightarrow B^s_{pq}(\mathbb{R}^n), \quad 0 < p \le \infty,\ 0 < q \le \infty,\ \sigma_p < s < u, \tag{4.34}$$

$$\mathrm{ext}_u : F^s_{pq}(\Omega) \hookrightarrow F^s_{pq}(\mathbb{R}^n), \quad 0 < p < \infty,\ 0 < q \le \infty,\ \sigma_{pq} < s < u. \tag{4.35}$$

Then there is for any ε with $0 < \varepsilon < 1$ a common extension operator ext^ε with

$$\mathrm{ext}^\varepsilon : B^s_{pq}(\Omega) \hookrightarrow B^s_{pq}(\mathbb{R}^n), \quad \begin{cases} \varepsilon < p \le \infty,\ 0 < q \le \infty,\ -\varepsilon^{-1} < s < 0, \\ 1 < p < \infty,\ 0 < q \le \infty,\ 0 \le s < \frac{n-d}{p}, \end{cases} \tag{4.36}$$

$$\operatorname{ext}^{\varepsilon}: F^s_{pq}(\Omega) \hookrightarrow F^s_{pq}(\mathbb{R}^n), \quad \begin{cases} \varepsilon < p < \infty, \ \varepsilon < q \le \infty, \ -\varepsilon^{-1} < s < 0, \\ 1 < p < \infty, \ 1 \le q < \infty, \ 0 \le s < \frac{n-d}{p}. \end{cases} \tag{4.37}$$

Proof. As mentioned in Section 4.1.1, part (i) goes back to [Ry99]. Theorem 4.10 covers the assertions about $\operatorname{ext}^{\varepsilon}$. By Proposition 3.18 with $\lambda = n - d > 0$ one has $|\Gamma| = 0$. Then one can apply Theorem 4.4. This covers the assertions about ext_u. □

Corollary 4.13. *Corollary* 4.12 (ii) *applies to the snowflake domain in Figure* 3.5*, p.* 76*, in* $\mathbb{R}^2$ *with* $d = \frac{\log 4}{\log 3}$.

Proof. This follows from Proposition 3.8. □

Remark 4.14. A first discussion of the extension problem for function spaces with positive smoothness on rough domains (beyond Lipschitz) has been given in Remark 3.3. There one finds also some references which will not be repeated here. But we are not aware of related assertions in the literature for spaces with negative smoothness as considered in Theorems 4.7, 4.10 and Corollary 4.12. However we wish to mention [Miy90] dealing in detail with the extension problem for Hardy spaces $H_p \sim F^0_{p,2}$ with $p < 1$ in arbitrary domains. The question of extendability is closely connected with the problem of intrinsic (quasi-)norms and related characterisations. In case of the classical Sobolev spaces one may consult Remark 2.54. As far as spaces on Lipschitz domains are concerned we refer to [T06], Sections 1.11, 1.11.10. But also some of the papers in Remark 3.3 deal with descriptions of function spaces of positive smoothness in more general domains. We mention in this context [See89] and the most recent substantial paper [Shv06]. In both papers the intrinsic characterisations of the spaces $A^s_{pq}(\Omega)$ are used to prove the existence of extension operators in $A^s_{pq}(\mathbb{R}^n)$. The admitted types of domains are similar to the I-thick domains according to Definition 3.1 (iii). Our way is different and resulted in Theorem 4.4 in a short, but less constructive proof of the existence of extension operators. In this book we do not deal with equivalent intrinsic quasi-norms in general. But we return to this point briefly in Section 4.3.3 and, as far as classical Sobolev spaces are concerned, in Theorem 4.30.

4.3 Complements

4.3.1 Interpolation

Interpolation is only a marginal topic in this book. Mainly we wish to demonstrate how overlapping regions of common extension operators, for example ext_u and $\operatorname{ext}^{\varepsilon}$ in Corollary 4.12, can be used to prove far-reaching interpolation results.

We assume that the reader is familiar with the basic assertions of interpolation theory. Let $\{A_0, A_1\}$ be an interpolation couple of complex quasi-Banach spaces. Then

$$(A_0, A_1)_{\theta,q}, \quad 0 < \theta < 1, \ 0 < q \le \infty, \tag{4.38}$$

denotes, as usual, the *real interpolation method* based on Peetre's K-functional. Let

$$[A_0, A_1]_\theta, \quad 0 < \theta < 1, \tag{4.39}$$

be the classical *complex interpolation method* for complex Banach spaces. Basic assertions about these interpolation methods may be found in [T78] and [BeL76]. One may also consult [T92], Section 1.6.

Proposition 4.15. *Let I be either the real method* (4.38) *or the complex method* (4.39). *Let $A_0(\mathbb{R}^n)$ and $A_1(\mathbb{R}^n)$ be two admitted spaces according to Definition* 1.1 *such that*

$$I(A_0(\mathbb{R}^n), A_1(\mathbb{R}^n)) = A(\mathbb{R}^n) \tag{4.40}$$

results in a space $A(\mathbb{R}^n)$ again covered by Definition 1.1. *Let Ω be a bounded Lipschitz domain according to Definition* 3.4 (iii) *in $\mathbb{R}^n$, $n \geq 2$, or an interval in $\mathbb{R}$. Let $A_0(\Omega)$, $A_1(\Omega)$, $A(\Omega)$ be the corresponding spaces on Ω as introduced in Definition* 2.1 (i). *Then*

$$I(A_0(\Omega), A_1(\Omega)) = A(\Omega). \tag{4.41}$$

Remark 4.16. This coincides essentially with [T06], Section 1.11.8. There one finds also a proof, further explanations and examples. In particular the classical complex interpolation method in (4.39) which is naturally restricted to Banach spaces can be extended to a class of quasi-Banach spaces which covers the spaces $A^s_{pq}(\Omega)$. Then (4.41) as a consequence of (4.40) remains valid also for this extended complex interpolation method which goes back to [MeM00]. Further information may also be found in [Tri02]. The proof that one can transfer (4.40) from $\mathbb{R}^n$ to (4.41) in bounded Lipschitz domains Ω relies on the existence of a universal extension operator as indicated in Corollary 4.12 (i). This is not available in more general domains. But the method works for those domains Ω and those spaces, say, $A^s_{pq}(\Omega)$ or $\tilde{A}^s_{pq}(\Omega)$, for which one has common extension operators from Ω to $\mathbb{R}^n$. Even more, if two (s, p, q)-regions in which one has common extension operators for some spaces $A^s_{pq}(\Omega)$ or $\tilde{A}^s_{pq}(\Omega)$ have a non-empty overlap then one can combine related interpolation formulas. We describe a typical example which makes clear what is meant and how to proceed in other cases. We rely on the common extension operators in Corollary 4.12 (ii).

Theorem 4.17. *Let Ω be a* (*non-empty*) *thick domain in $\mathbb{R}^n$ according to Definition* 3.1 (iv) *with $\bar{\Omega} \neq \mathbb{R}^n$ such that its boundary $\Gamma = \partial\Omega$ is an d-set satisfying* (4.29) *with $n-1 \leq d < n$* (*in case of $n = 1$ and $d = 0$ this means that Ω is an interval*). *Let $1 < p < \infty$ and $0 < \theta < 1$.*

(i) *Let $q_0, q_1, q \in (0, \infty]$. Let $-\infty < s_0 < s_1 < \infty$ and*

$$s = (1-\theta)s_0 + \theta s_1. \tag{4.42}$$

Then

$$\left(B^{s_0}_{pq_0}(\Omega), B^{s_1}_{pq_1}(\Omega)\right)_{\theta,q} = \left(F^{s_0}_{pq_0}(\Omega), F^{s_1}_{pq_1}(\Omega)\right)_{\theta,q} = B^s_{pq}(\Omega). \tag{4.43}$$

(ii) *Let* $q_0, q_1 \in (1,\infty)$, $s_0 \in \mathbb{R}$, $s_1 \in \mathbb{R}$, *and*

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{4.44}$$

Let s be given by (4.42). *Then*

$$\left[B^{s_0}_{pq_0}(\Omega), B^{s_1}_{pq_1}(\Omega)\right]_\theta = B^s_{pq}(\Omega) \tag{4.45}$$

and

$$\left[F^{s_0}_{pq_0}(\Omega), F^{s_1}_{pq_1}(\Omega)\right]_\theta = F^s_{pq}(\Omega). \tag{4.46}$$

Proof. Step 1. Since

$$B^s_{p,\min(p,q)}(\Omega) \hookrightarrow F^s_{pq}(\Omega) \hookrightarrow B^s_{p,\max(p,q)}(\Omega) \tag{4.47}$$

it is sufficient to prove (4.43) for the B-spaces. By [T83], Theorem 2.4.2, one has

$$\left(B^{s_0}_{pq_0}(\mathbb{R}^n), B^{s_1}_{pq_1}(\mathbb{R}^n)\right)_{\theta,q} = B^s_{pq}(\mathbb{R}^n). \tag{4.48}$$

We put temporarily

$$\left(B^{s_0}_{pq_0}(\Omega), B^{s_1}_{pq_1}(\Omega)\right)_{\theta,q} = B_\theta(\Omega). \tag{4.49}$$

Let $0 < s_0 < s_1 < u$. Then we can apply ext_u in (4.34) to all three spaces in (4.49) where we used the interpolation property as far as $B_\theta(\Omega)$ is concerned, hence

$$\mathrm{ext}_u \colon B_\theta(\Omega) \hookrightarrow B^s_{pq}(\mathbb{R}^n). \tag{4.50}$$

Let re be again the restriction operator, hence $\mathrm{re} \circ \mathrm{ext}_u = \mathrm{id}$. Then one obtains by (4.50) that

$$\mathrm{id} \colon B_\theta(\Omega) \hookrightarrow B^s_{pq}(\Omega). \tag{4.51}$$

Conversely, using (4.48) and the interpolation property applied to re one obtains

$$\mathrm{id} \colon B^s_{pq}(\Omega) \hookrightarrow B^s_{pq}(\mathbb{R}^n) \hookrightarrow B_\theta(\Omega). \tag{4.52}$$

Then (4.51), (4.52) prove (4.43) under the assumption $0 < s_0 < s_1$.

Step 2. Based on (4.36) one has by the same arguments

$$\left(B^{s_0}_{pq_0}(\Omega), B^{s_1}_{pq_1}(\Omega)\right)_{\theta,q} = B^s_{pq}(\Omega) \tag{4.53}$$

if $-\infty < s_0 < s_1 < \frac{n-d}{p}$. Note that

$$\left\{\left(\tfrac{1}{p}, s\right) : 1 < p < \infty,\ 0 < s < \tfrac{n-d}{p}\right\} \tag{4.54}$$

is a non-empty overlap of the admitted (p,s)-region with the corresponding one from Step 1. Then one can apply Wolff's interpolation theorem in [Wol82]. To extend (4.43) to all admitted p, q, s in part (i) of the theorem one may assume that $q_0, q_1 \in (1,\infty)$ or even $q_0 = q_1 = p$. Otherwise one could use afterwards the reiteration theorem of interpolation theory and what has been said so far to extend (4.43) from $q_0, q_1 \in (1,\infty)$ or $q_0 = q_1 = p$ to all $q_0, q_1 \in (0,\infty]$. In particular we may assume that $S(\Omega) = S(\mathbb{R}^n)|\Omega$, the restriction of $S(\mathbb{R}^n)$ to Ω, is dense in the spaces considered. Let for fixed p with $1 < p < \infty$,

$$A_j = B^{s_j}_{pp}(\Omega), \quad -\infty < s_1 < 0 < s_2 < s_3 < \frac{n-d}{p} < s_4 < \infty, \tag{4.55}$$

such that

$$A_2 = (A_1, A_3)_{\theta_1,p}, \quad A_3 = (A_2, A_4)_{\theta_2,p}, \tag{4.56}$$

covered by the two interpolation regions above. Then one obtains by [Wol82] that

$$A_2 = (A_1, A_4)_{\theta_3,p}, \quad A_3 = (A_1, A_4)_{\theta_4,p} \tag{4.57}$$

for naturally calculated $0 < \theta_3 < \theta_4 < 1$. This proves (4.43) for the full range of the admitted parameters.

Step 3. Similarly one can argue for the complex interpolation formulas using a corresponding assertion for the classical complex interpolation method in [Wol82]. For a more general version of Wolff's interpolation assertion one may also consult [JNP83]. □

Remark 4.18. To fix p in the above theorem is convenient but not necessary. Any interpolation formula in $\mathbb{R}^n$, real, classical complex, or generalised complex as in [MeM00], can be transferred from $\mathbb{R}^n$ to Ω as long as one has a common extension operator. This applies to the admitted (p,s)-regions in Step 1 and Step 2 based on Corollary 4.12. Similarly one can rely on Theorems 4.4, 4.7 and their Corollaries 4.5, 4.8. However if one wishes to clip together two such admitted (p,s)-regions for some spaces $A^s_{pq}(\Omega)$ then one needs an overlap as in (4.54) and that the desired interpolation fits in the scheme of (4.56), (4.57). We describe a further scale of spaces to which the above method can be applied.

There are several good reasons to ask for a counterpart of Theorem 4.17 with $\bar{B}^s_{pq}(\Omega)$ and $\bar{F}^s_{pq}(\Omega)$ as introduced in Definition 3.11 in place of $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$. On the one hand we have the common wavelet characterisations in Theorems 3.13, 3.23. On the other hand for bounded C^∞ domains Ω and fixed $1 < p < \infty$, $1 \le q \le \infty$,

$$\left\{\bar{B}^s_{pq}(\Omega) : s \in \mathbb{R}\right\} \quad \text{and} \quad \left\{\bar{H}^s_p(\Omega) = \bar{F}^s_{p,2}(\Omega) : s \in \mathbb{R}\right\} \tag{4.58}$$

are scales with lifting operators imitating I_σ in (1.15) for spaces on $\mathbb{R}^n$. This can be found in [T78], Section 4.9.2. There is also a related natural duality theory. An extension of these assertions, including duality and interpolation, to bounded Lipschitz domains has been given in [Tri02], Section 3.3 (there are also a few relevant comments in [T06], Section 1.11.6). Now we apply the above interpolation method to the corresponding spaces. As in Theorem 4.17 we restrict ourselves to typical examples.

Theorem 4.19. *Let Ω be a (non-empty) E-thick domain in $\mathbb{R}^n$ according to Definition 3.1* (ii) *with $\Omega \neq \mathbb{R}^n$ such that its boundary $\Gamma = \partial\Omega$ is an d-set satisfying* (4.29) *with* $n-1 \le d < n$ *(in case of* $n=1$ *and* $d=0$ *this means that* Ω *is an interval on* $\mathbb{R}$*). Let* $1<p<\infty$ *and* $0<\theta<1$*. Let* $\overline{B}^s_{pq}(\Omega)$ *and* $\overline{F}^s_{pq}(\Omega)$ *(now with* $\overline{F}^0_{pq}(\Omega) = F^0_{pq}(\Omega)$ *for all* $0<q\le\infty$*) be as in Definition* 3.11.

(i) *Let* $q_0, q_1, q \in (0,\infty]$*. Let* $-\infty < s_0 < s_1 < \infty$ *and*

$$s = (1-\theta)s_0 + \theta s_1. \tag{4.59}$$

Then

$$\big(\overline{B}^{s_0}_{pq_0}(\Omega), \overline{B}^{s_1}_{pq_1}(\Omega)\big)_{\theta,q} = \big(\overline{F}^{s_0}_{pq_0}(\Omega), \overline{F}^{s_1}_{pq_1}(\Omega)\big)_{\theta,q} = \overline{B}^s_{pq}(\Omega). \tag{4.60}$$

(ii) *Let* $q_0, q_1 \in (1,\infty)$*,* $s_0 \in \mathbb{R}$*,* $s_1 \in \mathbb{R}$ *and*

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{4.61}$$

Let s *be as in* (4.59)*. Then*

$$\big[\overline{B}^{s_0}_{pq_0}(\Omega), \overline{B}^{s_1}_{pq_1}(\Omega)\big]_\theta = \overline{B}^s_{pq}(\Omega) \tag{4.62}$$

and

$$\big[\overline{F}^{s_0}_{pq_0}(\Omega), \overline{F}^{s_1}_{pq_1}(\Omega)\big]_\theta = \overline{F}^s_{pq}(\Omega). \tag{4.63}$$

Proof. By Proposition 3.18 with $\lambda = n-d$ the domain Ω is uniformly E-porous with $|\Gamma| = 0$. We follow the arguments from the proof of Theorem 4.17. Let $0<s_0<s_1$. We have (4.48) and replace (as a definition) B in (4.49) by $\overline{B}$. By Proposition 3.15 one can identify $\overline{B}^s_{pq}(\Omega) = \widetilde{B}^s_{pq}(\Omega)$ with $\widetilde{B}^s_{pq}(\overline{\Omega})$. Then the extension ext by zero as in (3.89) is a linear and bounded extension operator from $\overline{B}^s_{pq}(\Omega)$ into $B^s_{pq}(\mathbb{R}^n)$ and we have (4.50) with ext in place of ext_u and $\overline{B}_\theta(\Omega)$. With the projection P as in Proposition 3.15 we have $\text{re} \circ P \circ \text{ext} = \text{id}$. Now one can argue as in (4.51), (4.52). This proves (4.59), (4.60) under the assumption $0<s_0<s_1$. By Theorem 4.10 we have (4.53) for $-\infty < s_0 < s_1 < \frac{n-d}{p}$. We can apply Proposition 3.19. This gives as before the desired overlap (4.54). The rest is now the same as in Steps 2 and 3 in the proof of Theorem 4.17. □

Corollary 4.20. *Theorems* 4.17*,* 4.19 *apply to bounded Lipschitz domains in* $\mathbb{R}^n$ *with* $n \ge 2$ *according to Definition* 3.4 (iii)*, to an interval on* $\mathbb{R}$*, and to the snowflake domain in Figure* 3.5*, p.* 76.

Proof. This follows from Proposition 3.8 with $d = \frac{\log 4}{\log 3}$ in case of the snowflake curve. □

4.3.2 Constrained wavelet expansions in Lipschitz domains

All wavelet expansions for function spaces in domains Ω considered so far are based on the u-wavelet systems $\{\Phi^j_r\}$ according to Definition 2.4. The building blocks Φ^j_r

have compact supports in Ω. This suggests that one cannot expect wavelet expansions based on $\{\Phi_r^j\}$ for spaces $A^s_{pq}(\Omega)$ having boundary values on $\Gamma = \partial\Omega$. Then one needs wavelets at the boundary in addition. We return to this problem in Chapter 5 below. On the other hand we described in Section 2.5.3 for Sobolev spaces $W^k_p(\Omega)$ in arbitrary domains so-called constrained wavelet expansions. These considerations relied on the specific structure of the norm in (2.222) reducing the question of whether $f \in W^k_p(\Omega)$ to $D^\alpha f \in L_p(\Omega)$ with $|\alpha| \le k$. One may ask for counterparts in case of the spaces $A^s_{pq}(\Omega)$ which can be used to argue as in Section 2.5.3. In arbitrary domains there is little hope for such a reduction. But the situation is much better if Ω is a Lipschitz domain.

Proposition 4.21. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \ge 2$ according to Definition 3.4* (iii) *or an interval on $\mathbb{R}$. Let $A^s_{pq}(\Omega)$,*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s = \sigma + k \text{ with } \sigma \in \mathbb{R} \text{ and } k \in \mathbb{N} \tag{4.64}$$

($p < \infty$ for the F-spaces), be the spaces as introduced in Definition 2.1. Then

$$A^s_{pq}(\Omega) = \left\{ f \in A^\sigma_{pq}(\Omega) : \; D^\alpha f \in A^\sigma_{pq}(\Omega), \; |\alpha| \le k \right\} \tag{4.65}$$

and

$$\|f \,|A^s_{pq}(\Omega)\| \sim \sum_{|\alpha| \le k} \|D^\alpha f \,|A^\sigma_{pq}(\Omega)\| \tag{4.66}$$

(equivalent quasi-norms).

Proof. Recall that for $g \in A^s_{pq}(\mathbb{R}^n)$,

$$\|g \,|A^s_{pq}(\mathbb{R}^n)\| \sim \sum_{|\alpha| \le k} \|D^\alpha g \,|A^\sigma_{pq}(\mathbb{R}^n)\| \tag{4.67}$$

(equivalent quasi-norms). We refer to [T83], Theorem 2.3.8, pp. 58–59. Then one obtains by Definition 2.1 that

$$\sum_{|\alpha| \le k} \|D^\alpha f \,|A^\sigma_{pq}(\Omega)\| \le c \, \|f \,|A^s_{pq}(\Omega)\| \tag{4.68}$$

for some $c > 0$ and all $f \in A^s_{pq}(\Omega)$. As for the converse we rely on Rychkov's universal extension operator ext_Ω according to [Ry99], Theorem 4.2, p. 253. By the convolution structure of this operator it follows from

$$f \in A^s_{pq}(\Omega) \iff g = \mathrm{ext}_\Omega \, f \in A^s_{pq}(\mathbb{R}^n) \tag{4.69}$$

that

$$D^\alpha f \in A^\sigma_{pq}(\Omega) \iff D^\alpha g = \mathrm{ext}_\Omega \, D^\alpha f \in A^\sigma_{pq}(\mathbb{R}^n), \tag{4.70}$$

where $0 \le |\alpha| \le k$. In particular one obtains by

$$\|f \,|A^s_{pq}(\Omega)\| \sim \|g \,|A^s_{pq}(\mathbb{R}^n)\| \sim \sum_{|\alpha| \le k} \|D^\alpha g \,|A^\sigma_{pq}(\mathbb{R}^n)\| \sim \sum_{|\alpha| \le k} \|D^\alpha f \,|A^\sigma_{pq}(\Omega)\| \tag{4.71}$$

the converse of (4.68) and the characterisation (4.65), (4.66). □

Remark 4.22. By (4.67) the counterpart of (4.66) with $\mathbb{R}^n$ in place of Ω is known. Furthermore characterisations of type (4.65), (4.66) for the half-space

$$\mathbb{R}^n_+ = \big\{x \in \mathbb{R}^n : \; x = (x', x_n), \; x' = (x_1, \ldots, x_{n-1}), \; x_n > 0\big\} \tag{4.72}$$

in place of Ω and for bounded C^∞ domains Ω according to Definition 3.4 (iii) may be found in [T83], Section 3.3.5, pp. 202–03. The arguments used there are similar as in (4.70). The above proposition remains valid for special Lipschitz domains according to Definition 3.4 (ii).

After these preparations we are now in a similar position as in Section 2.5.3 where we dealt with constrained wavelet expansions for Sobolev spaces in arbitrary domains. Instead of $L_p(\Omega)$ as there we use now

$$A^\sigma_{pq}(\Omega), \quad 0 < p \le \infty, \; 0 < q \le \infty, \; \sigma < 0, \tag{4.73}$$

($p < \infty$ for the F-spaces) in bounded Lipschitz domains Ω as basic spaces. By Proposition 3.8 bounded Lipschitz domains are thick and hence E-thick. Then we can apply Theorem 3.13 to the spaces in (4.73). Let

$$\Phi = \big\{\Phi^j_r : \; j \in \mathbb{N}_0; \; r = 1, \ldots, N_j\big\} \quad \text{with } N_j \in \mathbb{N}, \tag{4.74}$$

be the same orthonormal u-wavelet basis in $L_2(\Omega)$ according to Theorem 2.33 and Definitions 2.31, 2.4 as in (3.47) with

$$\sigma_p + |\sigma| < u \in \mathbb{N} \quad \text{and} \quad \sigma_{pq} + |\sigma| < u \in \mathbb{N}, \tag{4.75}$$

for the B-spaces and F-spaces, respectively. Let $b^s_{pq}(\mathbb{Z}_\Omega)$ and $f^s_{pq}(\mathbb{Z}_\Omega)$ be the same sequence spaces as in Definition 2.6 and as used in Theorem 3.13 now abbreviated by $a^s_{pq}(\mathbb{Z}_\Omega)$ with $a \in \{b, f\}$. Combining Proposition 4.21 with Theorem 3.13 one can argue as in Section 2.5.3. In particular if

$$f \in A^s_{pq}(\Omega), \quad s = \sigma + k, \; \sigma < 0, \; k \in \mathbb{N}, \tag{4.76}$$

then $D^\alpha f \in A^\sigma_{pq}(\Omega)$, $|\alpha| \le k$, can be expanded by

$$D^\alpha f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r(D^\alpha f)\, 2^{-jn/2}\, \Phi^j_r \tag{4.77}$$

where

$$\lambda^j_r(D^\alpha f) = 2^{jn/2}\,(D^\alpha f, \Phi^j_r) = (-1)^{|\alpha|}\, 2^{jn/2}\,(f, D^\alpha \Phi^j_r) \tag{4.78}$$

with

$$\lambda^j_r(f) = 2^{jn/2}\,(f, \Phi^j_r) = 2^{jn/2} \int_\Omega f(x)\, \Phi^j_r(x)\, \mathrm{d}x \tag{4.79}$$

as before (appropriately interpreted). Similarly as in (2.232), (2.233), we put

$$\lambda(f)^k = \big\{\lambda^j_r(f)^k \in \mathbb{R}_+ : \; j \in \mathbb{N}_0; \; r = 1, \ldots, N_j\big\} \tag{4.80}$$

with

$$\lambda_r^j(f)^k = 2^{jn/2} \sum_{|\alpha|\le k} \big|\big(f, D^\alpha \Phi_r^j\big)\big|. \tag{4.81}$$

Theorem 4.23. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \ge 2$ according to Definition 3.4* (iii) *or a bounded interval on $\mathbb{R}$. Let $A^s_{pq}(\Omega)$,*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s = \sigma + k \text{ with } \sigma < 0 \text{ and } k \in \mathbb{N}, \tag{4.82}$$

($p < \infty$ for the F-spaces) be the spaces as introduced in Definition 2.1. Let Φ be the orthonormal u-wavelet basis in $L_2(\Omega)$ with (4.74), (4.75). Then $f \in D'(\Omega)$ is an element of $A^s_{pq}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty}\sum_{r=1}^{N_j} \lambda_r^j(f)\, 2^{-jn/2}\, \Phi_r^j, \quad \lambda(f)^k \in a^\sigma_{pq}(\mathbb{Z}_\Omega). \tag{4.83}$$

Furthermore,

$$\| f \,|A^s_{pq}(\Omega)\| \sim \|\lambda(f)^k \,|a^\sigma_{pq}(\mathbb{Z}_\Omega)\| \tag{4.84}$$

(equivalent quasi-norms).

Proof. If $f \in D'(\Omega)$ can be represented by (4.83) then it follows from Theorem 3.13 that $D^\alpha f \in A^\sigma_{pq}(\Omega)$ for $|\alpha| \le k$. One has by Proposition 4.21 that $f \in A^s_{pq}(\Omega)$. Conversely if $f \in A^s_{pq}(\Omega)$ then $D^\alpha f$ with $|\alpha| \le k$ can be expanded in $A^\sigma_{pq}(\Omega)$ by (4.77), (4.78). Then (4.83), (4.84) follows from Proposition 4.21. □

Remark 4.24. As in Theorem 3.13 the representation (4.83) converges unconditionally in $A^\delta_{pq}(\Omega)$ with $\delta < \sigma$ and if, in addition, $p < \infty$, $q < \infty$ in $A^\sigma_{pq}(\Omega)$. Since the coefficients $\lambda_r^j(f)$ are subject to the *constraints* $\lambda(f)^k \in a^\sigma_{pq}(\mathbb{Z}_\Omega)$ it follows that the outcome belongs even to $A^s_{pq}(\Omega)$ with (4.84). There is the somewhat curious possibility to reduce the coefficients in (4.78) to linear combinations of the original coefficients in (4.79) at least if u in (4.75) is sufficiently large. Let again p, q, s be as in (4.82). Let $u \ge k$ in addition to (4.75). Then $D^\alpha \Phi_r^j$ with $|\alpha| \le k$ are continuous functions with compact supports in Ω. They can be expanded, say, in $L_2(\Omega)$ by

$$D^\alpha \Phi_r^j = \sum_{l=0}^{\infty}\sum_{t=1}^{N_l} \big(D^\alpha \Phi_r^j, \Phi_t^l\big)\, \Phi_t^l. \tag{4.85}$$

Inserted in (4.78) one obtains

$$\begin{aligned} \lambda_r^j(D^\alpha f) &= (-1)^{|\alpha|}\, 2^{jn/2} \sum_{l=0}^{\infty}\sum_{t=1}^{N_l} \big(D^\alpha \Phi_r^j, \Phi_t^l\big)\,(f, \Phi_t^l) \\ &= (-1)^{|\alpha|}\, 2^{(j-l)n/2} \sum_{l=0}^{\infty}\sum_{t=1}^{N_l} \big(D^\alpha \Phi_r^j, \Phi_t^l\big)\, \lambda_t^l(f) \\ &= \sum_{l=0}^{\infty}\sum_{t=1}^{N_l} m^\alpha_{(j,r;l,t)}\, \lambda_t^l(f) \end{aligned} \tag{4.86}$$

where

$$M_\Phi^\alpha = \{m^\alpha_{(j,r;l,t)}\}, \quad m^\alpha_{(j,r;l,t)} = (-1)^{|\alpha|}\, 2^{(j-l)n/2} \big(D^\alpha \Phi^j_r, \Phi^l_t\big) \tag{4.87}$$

are the *transition matrices* from Φ in (4.74) to

$$\Phi^\alpha = \big\{D^\alpha \Phi^j_r : j \in \mathbb{N}_0;\ r = 1, \ldots, N_j\big\}, \quad |\alpha| \le k. \tag{4.88}$$

Then (4.86) can be abbreviated in the usual way as

$$\lambda(D^\alpha f) = M_\Phi^\alpha \cdot \lambda(f), \quad |\alpha| \le k. \tag{4.89}$$

Corollary 4.25. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \ge 2$ according to Definition 3.4* (iii) *or a bounded interval on $\mathbb{R}$. Let $A^s_{pq}(\Omega)$ be the same spaces as in Theorem 4.23 with* (4.82)*. Let Φ be as above the orthonormal u-wavelet basis in $L_2(\Omega)$ with* (4.74), (4.75) *and $u \ge k$. Then $f \in D'(\Omega)$ is an element of $A^s_{pq}(\Omega)$ if, and only if, it can be represented by*

$$f = \sum_{j=0}^\infty \sum_{r=1}^{N_j} \lambda^j_r(f)\, 2^{-jn/2}\, \Phi^j_r \tag{4.90}$$

with

$$\sum_{|\alpha| \le k} \|M_\Phi^\alpha \cdot \lambda(f)\, | a^\sigma_{pq}(\mathbb{Z}_\Omega)\| < \infty \tag{4.91}$$

(*equivalent quasi-norms*).

Proof. This follows from Theorem 4.23 and the above considerations. □

Remark 4.26. At first glance the outcome is somewhat surprising. The entries of the transition matrices M_Φ^α in (4.87) depend only on the (sufficiently smooth) u-wavelet basis Φ. By the support properties of Φ^j_r the matrices M_Φ^α in (4.87) are band-limited. One needs only a knowledge of the coefficients $\lambda^j_r(f)$ in (4.79) to use the criterion (4.91) and to decide whether f belongs to $A^s_{pq}(\Omega)$ or not. If, say, $1 \le p < \infty$ and $s > 1/p$ then $f \in A^s_{pq}(\Omega)$ has boundary values at $\Gamma = \partial\Omega$. This seems to contradict the representation (4.90) with (4.91) since all building blocks Φ^j_r have compact supports in Ω and, hence, they vanish at Γ. But one must have in mind that (4.90) converges only in $A^\sigma_{pq}(\Omega)$ where $\sigma < 0$ (and, say, $p < \infty$, $q < \infty$). The constraint (4.91) does not improve this convergence. Furthermore the entries of the matrices M_Φ^α in (4.87) are real and (apparently) violently oscillating. Since Ω is bounded, (4.90) makes sense for any polynomial $f = P$. By (4.86) it follows that

$$M_\Phi^\alpha \cdot \lambda(P) = 0 \quad \text{if degree}(P) < |\alpha| \le k. \tag{4.92}$$

One has the impression that the terms with $|\alpha| > 0$ in (4.91) are the counterparts of the terms with $\Delta^m_{h,\Omega} f$ in (2.19) or corresponding terms involving oscillations as, for example, in [T92], Section 3.5. But so far it is not so clear what is the use of Theorem 4.23, Corollary 4.25 and also for the corresponding assertions for the classical Sobolev spaces in arbitrary domains in Section 2.5.3 (if there is any).

4.3.3 Intrinsic characterisations

According to *The New Shorter Oxford English Dictionary* on historical principles, Vol. 1 (Clarendon Press, Oxford, 1993), p. 1405, the word *intrinsic* has been in use in mathematics since the mid 19th century as *not involving reference to external coordinates*. In our context one may replace *coordinates* by *quantities* to make it sufficiently nebulous.

The classical Sobolev spaces $W^k_p(\Omega)$ with $1 < p < \infty$ and $k \in \mathbb{N}$ in arbitrary domains Ω are undoubtedly defined intrinsically by (2.221), (2.222). It is also acceptable to call the wavelet characterisations of these spaces in Theorem 2.55 intrinsic. On the other hand all spaces of type $A^s_{pq}(\Omega)$ we are dealing with in this exposition originate from $A^s_{pq}(\mathbb{R}^n)$ by restriction or decomposition. This cannot be called intrinsic with respect to a domain $\Omega \neq \mathbb{R}^n$. Then the question arises to find intrinsic descriptions where, according to the above quotation, the a somewhat vague notation *intrinsic* refers to, say, characterisations of these spaces on domains Ω where all ingredients (building blocks, quasi-norms etc.) are restricted to Ω. The orthonormal u-wavelet bases

$$\Phi = \left\{ \Phi^j_r : j \in \mathbb{N}_0;\ r = 1, \ldots, N_j \right\} \tag{4.93}$$

in $L_2(\Omega)$ according to Theorem 2.33 and Definitions 2.31, 2.4 fit in this scheme. This applies also to diverse sequence spaces of type $a^s_{pq}(\mathbb{Z}_\Omega)$ with $a \in \{b, f\}$. This justifies to call the following wavelet representations intrinsic.

(i) The characterisation of the refined localisation spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ in arbitrary domains Ω in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ according to Definition 2.14 in Theorem 3.27.

(ii) The characterisation of the spaces $\bar{A}^s_{pq}(\Omega)$ in E-thick domains Ω with $\Omega \neq \mathbb{R}^n$ according to Definition 3.11 with $s \neq 0$ in Theorem 3.13.

(iii) The characterisation of the spaces $\bar{A}^s_{pq}(\Omega)$ in uniformly E-porous (and hence by Proposition 3.18 E-thick) domains Ω with $\Omega \neq \mathbb{R}^n$ according to Definitions 3.11, 3.16 in Theorem 3.23.

(iv) The characterisation of the spaces $A^s_{pq}(\Omega)$ in bounded Lipschitz domains according to Definition 2.1 (i) in Theorem 4.23.

Remark 4.27. These are far-reaching characterisations in terms of common intrinsic wavelet bases, wavelet isomorphisms (onto sequence spaces of type $a^s_{pq}(\mathbb{Z}_\Omega)$), and constrained wavelet expansions. Parts (i)–(iii) apply not only to bounded Lipschitz domains but also to some domains with fractal boundaries covered by Propositions 3.8, 3.18, for example the snowflake domain in Figure 3.5, p. 76. The search for intrinsic characterisations of function spaces has a long history. But despite the above quotation from the Oxford English Dictionary it remains a matter of taste what one accepts to be called *intrinsic*. We return to this point in Remark 4.29 below. To some extent intrinsic characterisations are related to the extension problem. But this does not mean that the existence of a linear and bounded extension operator results automatically in an intrinsic characterisation and vice versa. For example, ext_u in Theorem 4.4 originating from (4.10) is hardly more than a non-constructive existence proof. It cannot be used to find an intrinsic characterisation of corresponding spaces $A^s_{pq}(\Omega)$ in I-thick domains. Or? On the other hand intrinsic characterisations of function spaces have been used in literature to construct extension operators. Although this is not our subject here we add a few further comments and give some references.

Remark 4.28. As far as the classical Sobolev spaces $W^k_p(\Omega)$ are concerned we commented on the extension problem in Section 2.5.3 and in Remark 3.3. There one finds also the relevant literature. Both extension problems and intrinsic characterisations for the Sobolev spaces and the classical Besov spaces

$$B^s_{pq}(\Omega), \quad 1 \le p, q \le \infty, \ s > 0, \tag{4.94}$$

including the Hölder–Zygmund spaces, and some forerunners such as the Slobodeckij spaces $B^s_{pp}(\Omega)$, have been studied since the 1950s and early 1960s. In [HaT08], Note 4.6.1, p. 112, we tried to clarify the historical roots, including relevant references. This will not be repeated here. We only mention that detailed discussions and corresponding results may be found in the two outstanding Russian books [Nik77], [BIN75]. One may also consult [T78] as far as the situation in the middle of the 1970s is concerned. For bounded C^∞ domains Ω the extension problem had been solved for all spaces $A^s_{pq}(\Omega)$ around 1990. This applies also to satisfactory intrinsic descriptions of the corresponding spaces $B^s_{pq}(\Omega)$ with $s > \sigma_p$ and $F^s_{pq}(\Omega)$ with $s > \sigma_{pq}$ in terms of differences, means of differences and oscillations. It may be found in [T92]. The solution of the extension problem and intrinsic characterisations in terms of convolutions and local means for all spaces $A^s_{pq}(\Omega)$ in bounded Lipschitz domains Ω goes back to [Ry98], [Ry99]. Intrinsic characterisations both of $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ in bounded Lipschitz domains for $s > \sigma_p$ or $s > \sigma_{pq}$ in terms of differences and means of differences may be found in [T06], Theorem 1.118, p. 74, with references to [Dis03], [DeS93] for the B-spaces. A special case of these characterisations has been mentioned above in Remark 2.3. Intrinsic descriptions and extensions of (anisotropic) spaces $F^s_{pq}(\Omega)$ with $s > \sigma_{pq}$ in (anisotropic) (ε, δ)-domains as in Definition 3.1 (i) are the subject of [See89]. The most recent and also most advanced paper in this connection is [Shv06] where the author studies intrinsic characterisations and extensions for spaces

$$W^k_p, \quad F^s_{pq}, \quad B^s_{pq} \quad \text{with } 1 \le p \le \infty, \ 1 \le q \le \infty, \ s > 0, \ k \in \mathbb{N},$$

on n-sets in $\mathbb{R}^n$ as in (3.79) with $h(r) = r^n$.

Remark 4.29. In order to prepare what follows we first recall of what we said so far about intrinsic characterisations for spaces on domains. As mentioned above with a reference to [Dis03], [DeS93] one has for the spaces $B^s_{pq}(\Omega)$ with (2.17) or, more generally, (2.20), in bounded Lipschitz domains the equivalent quasi-norms (2.19). According to Theorem 2.18 the refined localisation spaces $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ with (2.69), in particular $s > \sigma_{pq}$, can be described in arbitrary domains by ball means of differences. As a special case we obtained in Corollary 2.20 a corresponding characterisation for the related Sobolev spaces $W_p^{k,\mathrm{rloc}}(\Omega)$. We return to this point below. Some discussions about the extendability of Sobolev spaces including relevant references may be found in Remark 3.3, complemented by Remark 3.9. A discussion about the extension problem in thick domains inclusively related references has been given in Remark 4.14. Finally we recall the rather general intrinsic characterisations of the spaces $A^s_{pq}(\Omega)$ in terms of atoms as it has been developed in [TrW96] and described in [ET96], Section 2.5. Let Ω be a bounded domain in $\mathbb{R}^n$ with $\Omega = (\bar{\Omega})^\circ$ (which means that Ω coincides with the interior of its closure as considered in connection with Proposition 3.6). Then Ω is called *interior regular* if there is a positive number c such that

$$|\Omega \cap Q| \geq c\,|Q| \tag{4.95}$$

for any cube Q centred at $\partial\Omega$ with side-length less than 1. It is called *exterior regular* if there is a positive number c such that for any cube Q centred at $\partial\Omega$ with side-length l less than 1 there exists a subcube Q^e with side-length cl and

$$Q^e \subset Q \cap (\mathbb{R}^n \setminus \bar{\Omega}). \tag{4.96}$$

The above domain Ω is called *regular* if it is both interior and exterior regular. These notation are similar but not identical with I-thick, E-thick and thick domains as introduced in Definition 3.1. Somewhat roughly the multiplication of the (s,p)-atoms in $\mathbb{R}^n$ according to Definition 1.5 with the characteristic function of Ω are called Ω-*(s,p)-atoms*. Similarly one adapts the sequence spaces a_{pq} in Definition 1.3 to Ω in the same way as in Definition 2.6, now denoted as $a_{pq}(\mathbb{Z}_\Omega)$. For details and more precise formulations we refer to [TrW96] and [ET96], Section 2.5.2. Then it makes sense to ask for intrinsic characterisations of the spaces $A^s_{pq}(\Omega)$ as introduced in Definition 2.1,

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r \, a_{jr}, \qquad \lambda = \{\lambda^j_r\} \in a_{pq}(\mathbb{Z}_\Omega) \tag{4.97}$$

with Ω-(s,p)-atoms a_{jr} and

$$\| f \,|A^s_{pq}(\Omega)\| \sim \inf \|\lambda \,|a_{pq}(\mathbb{Z}_\Omega)\| \tag{4.98}$$

where the infimum is taken over all representations (4.97). This is the question for an intrinsic Ω-counterpart of Theorem 1.7. The outcome is the following. Let σ_p, σ_{pq} be the same numbers as in (1.32). *There is a characterisation* (4.97)*,* (4.98)*,*

(a) *for the spaces*

$$B^s_{pq}(\Omega), \quad 0 < p \leq \infty,\ 0 < q \leq \infty,\ s > \sigma_p, \tag{4.99}$$

in all bounded domains Ω with $\Omega = (\bar{\Omega})^\circ$,

(b) *for the spaces*

$$B^s_{pq}(\Omega), \quad 0 < p \leq \infty,\ 0 < q \leq \infty,\ s \in \mathbb{R}, \tag{4.100}$$

in exterior regular domains,

(c) *for the spaces*

$$F^s_{pq}(\Omega), \quad 0 < p < \infty,\ 0 < q \le \infty,\ s > \sigma_{pq}, \tag{4.101}$$

in interior regular domains, and

(d) *for the spaces*

$$F^s_{pq}(\Omega), \quad 0 < p < \infty,\ 0 < q \le \infty,\ s \in \mathbb{R}, \tag{4.102}$$

in regular domains.

Greater details and further explanations may be found in [ET96], Section 2.5. As for proofs we refer to [TrW96]. Nowadays the arguments given there may be simplified a little bit (but presumably not very much) by using wavelet isomorphisms for the spaces $A^s_{pq}(\mathbb{R}^n)$ according to Theorem 1.20 as the starting point instead of atomic representations as described in Theorem 1.7. But there is a difference between the intrinsic wavelet characterisations as recalled in (i)–(iv) at the beginning of this section and the above descriptions (4.97), (4.98) for the spaces in (a)–(d). On the one hand the wavelet coefficients $\lambda^j_r(f)$ according to (3.127) are known (more or less) explicitly and one has to check whether $\{\lambda^j_r(f)\}$ belongs to some sequence spaces of type $a^s_{pq}(\mathbb{Z}_\Omega)$ or not. But there are no constructive counterparts for optimal coefficients and atoms in (4.97), (4.98) and one may even question whether these representations are entitled to be called intrinsic. But this is a little bit like the unsolvable riddle, who was first, the hen (H) or the egg (E)?

(H) *Given* $f \in D'(\Omega)$ *or* $f \in S'(\Omega) = S(\mathbb{R}^n)|\Omega$. *Decide intrinsically to which spaces* $A^s_{pq}(\Omega)$ *it belongs.*

(E) *Given* $A^s_{pq}(\Omega)$. *Find an intrinsic description of all of its elements.*

Functions have been for ages, are, and will be for ever the favourite subject of mathematicians independently of whether a few of them are recruited to serve as members of function spaces. The above wavelet characterisations (i)–(iv) fit both in (H) and (E) whereas (4.97), (4.98) in (a)–(d) are restricted to (E).

Finally we wish to demonstrate how wavelet bases can be used to obtain intrinsic equivalent norms for Sobolev spaces in E-thick domains which in the case of bounded smooth domains are more or less known. We introduce the notation

$$\widetilde{W}^k_p(\Omega) = \widetilde{F}^k_{p,2}(\Omega), \quad k \in \mathbb{N},\ 1 < p < \infty, \tag{4.103}$$

where $\widetilde{F}^k_{p,2}(\Omega)$ are the same spaces as in Definition 2.1 (ii). Let as in (2.67),

$$\delta(x) = \min(1, \operatorname{dist}(x, \Gamma)), \quad \Gamma = \partial\Omega,\ x \in \Omega. \tag{4.104}$$

Theorem 4.30. *Let* Ω *be an* E*-thick domain in* $\mathbb{R}^n$ *according to Definition* 3.1 (ii) *such that* $|\partial\Omega| = 0$ *and*

$$\left|\left\{x \in \Omega : \, \delta(x) \geq 2^{-j}\right\}\right| < \infty, \quad j \in \mathbb{N}. \tag{4.105}$$

Let $\widetilde{W}_p^k(\Omega)$ *with* $k \in \mathbb{N}$ *and* $1 < p < \infty$ *be as in* (4.103)*. Then* $D(\Omega)$ *is dense in* $\widetilde{W}_p^k(\Omega)$*, the embedding*

$$\mathrm{id}\colon \widetilde{W}_p^k(\Omega) \hookrightarrow L_p(\Omega) \tag{4.106}$$

is compact, and

$$\|f \,|\widetilde{W}_p^k(\Omega)\|_1 = \sum_{|\alpha|\leq k} \|\delta^{-k+|\alpha|} D^\alpha f \,|L_p(\Omega)\|, \tag{4.107}$$

$$\|f \,|\widetilde{W}_p^k(\Omega)\|_2 = \sum_{|\alpha|=k} \|D^\alpha f \,|L_p(\Omega)\| + \|\delta^{-k} f \,|L_p(\Omega)\|, \tag{4.108}$$

$$\|f \,|\widetilde{W}_p^k(\Omega)\|_3 = \sum_{|\alpha|\leq k} \|D^\alpha f \,|L_p(\Omega)\|, \tag{4.109}$$

$$\|f \,|\widetilde{W}_p^k(\Omega)\|_4 = \sum_{|\alpha|=k} \|D^\alpha f \,|L_p(\Omega)\|, \tag{4.110}$$

are equivalent norms in $\widetilde{W}_p^k(\Omega)$.

Proof. Step 1. By Proposition 3.10 and Corollary 2.20 it follows that (4.107) and (4.108) are equivalent norms on $\widetilde{W}_p^k(\Omega)$.

Step 2. By Theorems 3.13 and 2.36 one has for $\widetilde{W}_p^k(\Omega)$ and $L_p(\Omega)$ common wavelet bases $\{\Phi_r^j\}$ and related expansions

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda_r^j(f)\, 2^{-jn/2}\, \Phi_r^j \tag{4.111}$$

with

$$\|f \,|\widetilde{W}_p^k(\Omega)\| \sim \|\lambda \,|f_{p,2}^k(\mathbb{Z}_\Omega)\|, \quad \|f \,|L_p(\Omega)\| \sim \|\lambda \,|f_{p,2}^0(\mathbb{Z}_\Omega)\|. \tag{4.112}$$

The assumption (4.105) ensures $N_J < \infty$ for any $J \in \mathbb{N}$. In particular, id_J,

$$\mathrm{id}_J f = \sum_{j=0}^{J} \sum_{r=1}^{N_j} \lambda_r^j(f)\, 2^{-jn/2}\, \Phi_r^j, \tag{4.113}$$

is a linear operator of finite rank. By (2.38) one obtains

$$\|f - \mathrm{id}_J f \,|L_p(\Omega)\| \leq c\, 2^{-kJ}\, \|f \,|\widetilde{W}_p^k(\Omega)\|, \quad f \in \widetilde{W}_p^k(\Omega). \tag{4.114}$$

Now it follows by standard arguments that id in (4.106) is compact. Since each function Φ_r^j can be approximated, say, in the norm in (4.107) by functions belonging to $D(\Omega)$ (Sobolev's mollification method) one obtains as a by-product that $D(\Omega)$ is dense in $\widetilde{W}_p^k(\Omega)$.

Step 3. Since $|\partial\Omega| = 0$ one sees by Proposition 3.15 that one can identify $\widetilde{W}_p^k(\Omega)$ with $\widetilde{W}_p^k(\bar{\Omega})$, a subspace of $W_p^k(\mathbb{R}^n)$. This proves that both (4.109) and

$$\|f \,|\widetilde{W}_p^k(\Omega)\|_5 = \|f \,|L_p(\Omega)\| + \sum_{|\alpha|=k} \|D^\alpha f \,|L_p(\Omega)\| \tag{4.115}$$

are equivalent norms in $\widetilde{W}_p^k(\Omega)$. To show that even (4.110) is an equivalent norm it remains to prove that there is a positive constant c such that

$$\|f \,|L_p(\Omega)\| \le c \sum_{|\alpha|=k} \|D^\alpha f \,|L_p(\Omega)\| \quad \text{for all } f \in \widetilde{W}_p^k(\Omega). \tag{4.116}$$

This will be done by contradiction assuming that there is no positive constant c with (4.116). Then one finds a sequence $\{f_j\}_{j=1}^\infty \subset \widetilde{W}_p^k(\Omega)$ with

$$1 = \|f_j \,|L_p(\Omega)\| > j \sum_{|\alpha|=k} \|D^\alpha f_j \,|L_p(\Omega)\|. \tag{4.117}$$

In particular, $\{f_j\}$ is bounded in $\widetilde{W}_p^k(\Omega)$. Since id in (4.106) is compact we may assume that f_j converges in $L_p(\Omega)$,

$$f_j \to f \in L_p(\Omega), \quad \|f \,|L_p(\Omega)\| = 1. \tag{4.118}$$

Then it follows by (4.117) that $\{f_j\}$ converges even in $\widetilde{W}_p^k(\Omega)$ to f. By (4.117) one has $D^\alpha f = 0$ for $|\alpha| = k$. We may assume that Ω is connected. As a consequence, f is a polynomial of degree less than k in Ω,

$$f = \sum_{|\beta|\le k-1} a_\beta \, x^\beta \in \widetilde{W}_p^k(\Omega), \quad \|f \,|L_p(\Omega)\| = 1. \tag{4.119}$$

Applying again Proposition 3.15 one obtains

$$a_\beta \in \widetilde{W}_p^1(\Omega), \quad |\beta| = k-1. \tag{4.120}$$

For fixed $l \in \mathbb{N}$ the number of Whitney cubes Q_{lr}^0 needed in (2.83), (2.84) is larger than $c\, 2^{l(n-1)}$ for some $c > 0$ which is independent of $l \in \mathbb{N}$. This follows also from the arguments in Step 3 of the proof of Proposition 3.18. Then one has

$$\int_\Omega \delta^{-p}(x)\,\mathrm{d}x \ge c' \sum_{l=1}^\infty 2^{lp}\, 2^{l(n-1)}\, 2^{-ln} = \infty, \tag{4.121}$$

where we used $p > 1$. We apply (4.108) with $k = 1$ to (4.120) and obtain $a_\beta = 0$ for $|\beta| = k-1$. Iteration gives $a_\beta = 0$ for all $|\beta| \le k-1$. Hence $f = 0$. But this contradicts (4.119). □

4.3.4 Compact embeddings

If Ω is an E-thick domain in $\mathbb{R}^n$ with (4.105) then the embedding (4.106) is compact. This observation is based on the wavelet representations (4.111), (4.112) and the approximation of the identity id by the finite rank operators id_J in (4.114). If one knows the dimension of the range of id_J then one obtains by (4.114) an estimate of the so-called approximation numbers of id in (4.106). It is the main aim of what follows to have a closer look at this type of argument and to make clear what is new compared with already existing results. But we restrict ourselves to a few comments. First we recall some definitions and quote related assertions. Let

$$U_A = \{a \in A : \ \|a \,|A\| \le 1\} \tag{4.122}$$

be the unit ball in the (complex) quasi-Banach space A. If A, B are two (complex) quasi-Banach spaces then $L(A, B)$ is the collection of all linear and bounded operators $T : A \hookrightarrow B$ quasi-normed by

$$\|T\| = \sup\{\|Ta \,|B\| : \ a \in U_A\}. \tag{4.123}$$

Furthermore, rank T denotes the dimension of the range of T. Recall that $T \in L(A, B)$ is called *compact* if $T U_A$ is precompact in B.

Definition 4.31. Let A, B be quasi-Banach spaces and let $T \in L(A, B)$.

(i) Then for all $k \in \mathbb{N}$ the k-th entropy number $e_k(T)$ of T is defined as the infimum of all $\varepsilon > 0$ such that

$$T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon U_B) \quad \text{for some } b_1, \dots, b_{2^{k-1}} \in B. \tag{4.124}$$

(ii) Then for all $k \in \mathbb{N}$ the k-th approximation number $a_k(T)$ of T is defined by

$$a_k(T) = \inf\{\|T - L\| : \ L \in L(A, B), \ \text{rank } L < k\}. \tag{4.125}$$

Remark 4.32. Neither entropy numbers nor approximation numbers and their relations to the spectral theory of compact operators play any role in this book. This may justify that we refer to [T06], Section 1.10, for further comments, properties, and the relevant literature. Recall that we explained in (3.4), (3.5) what is meant by the equivalence $a_k \sim b_k$. Let $B^s_{pq}(\Omega)$ be the spaces as introduced in Definition 2.1 by restriction of $B^s_{pq}(\mathbb{R}^n)$ to Ω. Let $\frac{1}{p} + \frac{1}{p'} = 1$ if $1 \le p \le \infty$ and $p' = \infty$ if $0 < p < 1$. Recall that $b_+ = \max(b, 0)$ for $b \in \mathbb{R}$.

Theorem 4.33. (i) *Let Ω be an arbitrary bounded domain in $\mathbb{R}^n$. Let $p_0, p_1, q_0, q_1 \in (0, \infty]$ and*

$$-\infty < s_1 < s_0 < \infty, \quad s_0 - \frac{n}{p_0} > s_1 - \frac{n}{p_1}. \tag{4.126}$$

Then the embedding

$$\mathrm{id} : B^{s_0}_{p_0 q_0}(\Omega) \hookrightarrow B^{s_1}_{p_1 q_1}(\Omega) \tag{4.127}$$

is compact and

$$e_k(\mathrm{id}) \sim k^{-\frac{s_0-s_1}{n}}, \quad k \in \mathbb{N}. \tag{4.128}$$

(ii) *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$*,* $n \geq 2$*, according to Definition* 3.4 (iii) *or a bounded interval in* $\mathbb{R}$ *if* $n = 1$*. Let* $p_0, p_1, q_0, q_1, s_0, s_1$ *be as in part* (i) *and let*

$$\delta_+ = s_0 - s_1 - n\Big(\frac{1}{p_0} - \frac{1}{p_1}\Big)_+, \tag{4.129}$$

$$\lambda = \frac{s_0 - s_1}{n} - \max\Big(\frac{1}{2} - \frac{1}{p_1}, \frac{1}{p_0} - \frac{1}{2}\Big). \tag{4.130}$$

Let id *be as in* (4.127)*. Then for* $k \in \mathbb{N}$*,*

$$a_k(\mathrm{id}) \sim k^{-\delta_+/n} \quad \text{if} \begin{cases} \text{either} & 0 < p_0 \leq p_1 \leq 2, \\ \text{or} & 2 \leq p_0 \leq p_1 \leq \infty, \\ \text{or} & 0 < p_1 \leq p_0 \leq \infty, \end{cases} \tag{4.131}$$

$$a_k(\mathrm{id}) \sim k^{-\lambda} \quad \text{if } 0 < p_0 < 2 < p_1 < \infty,\ \lambda > 1/2, \tag{4.132}$$

and

$$a_k(\mathrm{id}) \sim k^{-\frac{\delta_+}{n}\cdot\frac{\min(p_0', p_1)}{2}} \quad \text{if } 0 < p_0 < 2 < p_1 < \infty,\ \lambda < 1/2. \tag{4.133}$$

Remark 4.34. All equivalences are independent of q_0 and q_1. Then it follows from

$$B^s_{p,\min(p,q)}(\Omega) \hookrightarrow F^s_{pq}(\Omega) \hookrightarrow B^s_{p,\max(p,q)}(\Omega) \tag{4.134}$$

that one can replace B in (4.127) by F or $A \in \{B, F\}$. Both (4.128) and the corresponding assertions for $a_k(\mathrm{id})$ have a long history. We only mention that for bounded C^∞ domains (4.128) goes back to [EdT89], [EdT92] and [ET96], Section 3.3, and (4.131), (4.132) to [ET96], Section 3.3, complemented in [Cae98] by (4.133). There one finds also the history and further references. As for more recent comments one may also consult [T06], Sections 1.11.2, 1.11.7. The main reason for incorporating the above theorem is the following. Both for entropy numbers and approximation numbers it is assumed that Ω is bounded; in case of the approximation numbers one relies in addition on Corollary 4.12 (i) ensuring that there is a common extension operator for the spaces on Ω into corresponding spaces on $\mathbb{R}^n$. Then multiplication with a suitable cut-off function shows that it is sufficient to deal with embeddings

$$\mathrm{id}_0 : \big\{ f \in B^{s_0}_{p_0 q_0}(\mathbb{R}^n) : \ \operatorname{supp} f \subset U \big\} \hookrightarrow B^{s_1}_{p_1 q_1}(\mathbb{R}^n) \tag{4.135}$$

where $U = \{y \in \mathbb{R}^n : |y| \leq 1\}$ is the unit ball in $\mathbb{R}^n$. Then one can apply the wavelet isomorphisms in $\mathbb{R}^n$ according to Theorem 1.20. This reduces the study of entropy numbers and approximation numbers for id in (4.127) or id_0 in (4.135) to equivalent problems for sequence spaces. This reduction is quite standard nowadays. It had been used for the first time in [T97] and afterwards in [T01], [T06] and the related literature mentioned there. We describe the outcome of this reduction and adapt the sequence spaces b^s_{pq} according to Definition 1.18 and Theorem 1.20 to the notation as used in [T97], [T01] and more recent in [T06], Section 6.3. Let

$$\lambda = \{\lambda_{jr} \in \mathbb{C} : j \in \mathbb{N}_0;\ r = 1, \ldots, M_j\}, \quad M_j \sim 2^{jd}, \tag{4.136}$$

for $d > 0$. With $\delta \geq 0$, $0 < p \leq \infty$, $0 < q \leq \infty$ let $\ell_q\big(2^{j\delta}\ell_p^{M_j}\big)$ be the space of all sequences λ in (4.136) such that

$$\big\| \lambda \,|\ell_q\big(2^{j\delta}\ell_p^{M_j}\big)\big\| = \bigg(\sum_{j=0}^{\infty} 2^{j\delta q} \Big(\sum_{r=1}^{M_j} |\lambda_{jr}|^p \Big)^{q/p} \bigg)^{1/q} < \infty, \tag{4.137}$$

(usual modification if $p = \infty$ and/or $q = \infty$). By Theorem 1.20 it is now quite clear that the embedding (4.135) and the study of related entropy and approximation numbers can be reduced to corresponding equivalent questions for related sequence spaces with $d = n$. All this has been done in detail in the above-mentioned literature. But this observation paves the way to apply the above common wavelet bases to study embeddings of type (4.127) now for more general situations and modified spaces. We restrict ourselves to a few examples and comments.

Theorem 4.35. *Let Ω be an E-thick domain in $\mathbb{R}^n$ according to Definition 3.1* (ii) *with $|\Omega| < \infty$. Let $p_0, p_1, q_0, q_1 \in (0, \infty]$ and*

$$-\infty < s_1 < s_0 < 0, \quad s_0 - \frac{n}{p_0} > s_1 - \frac{n}{p_1}. \tag{4.138}$$

Then the embedding

$$\mathrm{id}: B^{s_0}_{p_0q_0}(\Omega) \hookrightarrow B^{s_1}_{p_1q_1}(\Omega) \tag{4.139}$$

is compact. Furthermore one has (4.128) *for the related entropy numbers and* (4.131)–(4.133) *with δ_+ and λ as in* (4.129), (4.130) *for the related approximation numbers.*

Proof. Since $s_1 < s_0 < 0$ one can apply Theorem 3.13 (ii) to the two spaces in (4.139). By construction one has $N_j \sim 2^{jn}$. This reduces the spaces $b^s_{pq}(\mathbb{Z}_\Omega)$ in (3.53) to the above sequence spaces $\ell_q\big(2^{j\delta}\ell_p^{M_j}\big)$ with $M_j = N_j$ in the same way as outlined in Remark 4.34 in connection with id_0 in (4.135). Then one obtains the above theorem as a corollary of Theorem 4.33. □

Remark 4.36. The main point is the direct transfer of (4.139) to corresponding embeddings between sequence spaces by using the common wavelet basis according to Theorem 3.13. One needs that $|\Omega| < \infty$ but not that Ω is bounded. Under these circumstances a reduction of id in (4.139) to id_0 in (4.135) is not always possible. But this was essential for the proof of Theorem 4.33. By Proposition 3.8 the above theorem can be applied to the snowflake domain in $\mathbb{R}^2$ according to Figure 3.5, p. 76.

Remark 4.37. One has now two possibilities to extend the assertions about approximation numbers according to Theorem 4.33 (ii) from bounded Lipschitz domains to more general domains. Let id_0 be the same reference embedding as in (4.135). If Ω is bounded and if there exists a linear and continuous extension operator

$$\mathrm{ext}\colon B^{s_0}_{p_0 q_0}(\Omega) \hookrightarrow B^{s_0}_{p_0 q_0}(\mathbb{R}^n) \tag{4.140}$$

for the source space then one obtains by the arguments indicated in Remark 4.34 that

$$a_k(\mathrm{id}) \sim a_k(\mathrm{id}_0), \quad k \in \mathbb{N}, \tag{4.141}$$

where id is the same embedding as in (4.126), (4.127). Secondly one obtains the same result if $|\Omega| < \infty$ (but Ω not necessarily bounded) and if both the source space and the target space have a common wavelet basis. It seems to be reasonable to fix some assertions which can be obtained in this way as a corollary both to Theorem 4.33, Remark 4.34 on the one hand and Theorem 4.35 and its proof on the other hand. Let σ_p and σ_{pq} be as in (4.15). As before a_k are the approximation numbers according to (4.125).

Corollary 4.38. (i) *Let Ω be a bounded I-thick domain in $\mathbb{R}^n$ according to Definition 3.1 (iii) with $|\partial\Omega| = 0$. Let $p_0, q_0, p_1, q_1, s_0, s_1$ be as in Theorem 4.33 and let in addition $s_0 > \sigma_{p_0}$. Then* id *in (4.127) is compact with*

$$a_k(\mathrm{id}) \sim a_k(\mathrm{id}_0), \quad k \in \mathbb{N}, \tag{4.142}$$

where id_0 is the reference embedding (4.135).

(ii) *Let Ω be an E-thick domain in $\mathbb{R}^n$ according to Definition 3.1 (ii) with $|\Omega| < \infty$. Let $p_0, q_0, p_1, q_1, s_0, s_1$ be as in Theorem 4.33 and let in addition*

$$s_0 \notin [0, \sigma_{p_0}] \quad \text{and} \quad s_1 \notin [0, \sigma_{p_1}]. \tag{4.143}$$

Let $\bar{B}^s_{pq}(\Omega)$ be the spaces according to (3.46). Then

$$\mathrm{id}\colon \bar{B}^{s_0}_{p_0 q_0}(\Omega) \hookrightarrow \bar{B}^{s_1}_{p_1 q_1}(\Omega) \tag{4.144}$$

is compact with (4.142) where id_0 has the same meaning as there. One can replace

$$\bar{B}^{s_0}_{p_0 q_0}(\Omega) \quad \text{by} \quad L_{p_0}(\Omega) \quad \text{with } 1 < p_0 < \infty,\ -\frac{n}{p_0} > s_1 - \frac{n}{p_1}, \tag{4.145}$$

or

$$\bar{B}^{s_1}_{p_1 q_1}(\Omega) \quad \text{by} \quad L_{p_1}(\Omega) \quad \text{with } 1 < p_1 < \infty,\ s_0 - \frac{n}{p_0} > -\frac{n}{p_1}. \tag{4.146}$$

(iii) *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $|\Omega| < \infty$. Let p_0, q_0, p_1, q_1, s_0, s_1 be as in Theorem 4.33 and let in addition*

$$s_0 \notin [0, \sigma_{p_0 q_0}], \quad s_1 \notin [0, \sigma_{p_1 q_1}], \tag{4.147}$$

$q_0 = \infty$ if $p_0 = \infty$ and $q_1 = \infty$ if $p_1 = \infty$. Let $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ be the spaces according to Definition 2.14. Then

$$\mathrm{id}\colon\ F^{s_0,\mathrm{rloc}}_{p_0q_0}(\Omega) \hookrightarrow F^{s_1,\mathrm{rloc}}_{p_1q_1}(\Omega) \tag{4.148}$$

is compact with (4.142) *in the cases covered by Theorem* 4.33 (ii) *and* id_0 *as in* (4.135). *One can replace*

$$F^{s_0,\mathrm{rloc}}_{p_0q_0}(\Omega) \quad \textit{by} \quad L_{p_0}(\Omega) \quad \textit{with } 1 < p_0 < \infty,\ -\tfrac{n}{p_0} > s_1 - \tfrac{n}{p_1}, \tag{4.149}$$

or

$$F^{s_1,\mathrm{rloc}}_{p_1q_1}(\Omega) \quad \textit{by} \quad L_{p_1}(\Omega) \quad \textit{with } 1 < p_1 < \infty,\ s_0 - \tfrac{n}{p_0} > -\tfrac{n}{p_1}. \tag{4.150}$$

Proof. It is sufficient to complement the arguments in Remark 4.37 by the necessary references. Since Ω is bounded one obtains part (i) from Theorem 4.4 and (4.140), (4.141). Theorem 3.13 complemented by Theorem 2.36 ensure the existence of common wavelet bases in E-thick domains for the spaces considered. Since $|\Omega| < \infty$ one obtains part (ii) by the same arguments as in the proof of Theorem 4.35. Similarly for part (iii) where one can rely on Theorem 3.27. Then one needs the F-counterpart of (4.142). But by Remark 4.34 this is the same as for the B-spaces excluding limiting cases. □

Remark 4.39. By (4.134) one can incorporate the F-spaces in parts (i) and (ii). As far as spaces of smoothness zero are concerned we restricted the above formulation to $L_p(\Omega)$ with $1 < p < \infty$. This can be extended to some other spaces $A^0_{pq}(\Omega)$ under some restrictions for the parameters and domains. One may consult the theorems mentioned in the above proof.

Remark 4.40. As in Theorem 4.35 one can replace the approximation numbers a_k in the above corollary by the corresponding entropy numbers e_k with (4.128). This is of interest for parts (ii) and (iii) but not for part (i) where we have by Theorem 4.33 (i) a better assertion. Furthermore one can ask to which extent one can replace approximation numbers a_k and entropy numbers e_k by other s*-numbers* or diverse types of *widths* measuring compactness. In the first line one may think about *Kolmogorov numbers* and *Gelfand numbers*. But there is a plethora of other numbers and widths. The abstract theory of s-numbers and widths in Banach spaces may be found in [Pie80], Section 11, [Pie87], Chapter 2, [Pie07], Section 6.2, and [Kon86]. There are some specifications of the abstract theory to sequence spaces and function spaces. We refer in this context also to the recent report [Vyb07b], filling also some gaps as far as approximation numbers for compact embeddings between function spaces in bounded Lipschitz domains according to Theorem 4.33 (ii) are concerned. If one wishes to use what is known so far for some widths, denoted by $\{h_k : k \in \mathbb{N}\}$, for the above purposes then the following ingredients are desirable.

- The numbers $\{h_k(T)\}$ are defined for compact maps $T\colon A \hookrightarrow B$ between (complex) quasi-Banach spaces and

$$\|T\| \ge h_1(T) \ge \cdots \ge h_k(T) \ge \cdots \tag{4.151}$$

(monotonically decreasing).

- They have the multiplication property

$$h_k(T_2 \circ T \circ T_1) \le \|T_2\| \cdot h_k(T) \cdot \|T_1\|, \quad k \in \mathbb{N}, \tag{4.152}$$

where

$$T_1\colon A \hookrightarrow A_1, \quad T\colon A_1 \hookrightarrow B_1 \text{ compact}, \quad T_2\colon B_1 \hookrightarrow B. \tag{4.153}$$

- One knows $h_k(\mathrm{id}_0)$ for reference mappings of type (4.135),

$$\mathrm{id}_0\colon \big\{ f \in A^{s_0}_{p_0 q_0}(\mathbb{R}^n) : \operatorname{supp} f \subset U \big\} \hookrightarrow A^{s_1}_{p_1 q_1}(\mathbb{R}^n). \tag{4.154}$$

Afterwards one can switch from a counterpart of Theorem 4.33 (ii) with h_k in place a_k (if exists) to the corresponding counterparts of Theorem 4.35 and Corollary 4.38 by the same arguments as above.

Remark 4.41. So far we replaced the assumption that Ω is bounded in Theorem 4.33 by $|\Omega| < \infty$ in Theorem 4.35 and in the parts (ii) and (iii) of Corollary 4.38. Then one has the sequence spaces in (4.136), (4.137) with $d = n$. But there is an elaborated theory for compact embeddings between sequence spaces of this type for all $d > 0$ covering at least equivalence assertions for the corresponding entropy numbers. We refer to [T06], Section 6.3, and the literature given there. If, for example, Ω is E-thick with $|\Omega| = \infty$ and

$$\big| \big\{ x \in \Omega : \delta(x) \ge 2^{-j} \big\} \big| \sim 2^{\varepsilon j}, \quad j \in \mathbb{N}, \tag{4.155}$$

for some $\varepsilon > 0$, where $\delta(x)$ has the same meaning as in (4.104), then it seems to be possible to obtain equivalence assertions at least for the entropy numbers of id in (4.139). As far as sequence spaces are concerned there are far-reaching generalisations of the above sequence spaces of type (4.136), (4.137) for which one has equivalence assertions for entropy numbers of related compact embeddings. We refer to [T06], Remark 6.18, pp. 278–79, where we collected corresponding papers. It remains to be seen to which extent these results can be used in the above context. One may even ask for inverse assertions. For this purpose it might be useful to convert the decreasing sequence $\{e_k\}$ of entropy numbers of a compact embedding between two function spaces into a decreasing (which means non-increasing) function on $[0, \infty)$,

$$e(t) = e_k \quad \text{for } k - 1 \le t < k,\ k \in \mathbb{N}. \tag{4.156}$$

Of course, $e(t) \to 0$ if $t \to \infty$. Let $g(t)$ be a, say, continuous, positive decreasing function on $[0, \infty)$ with $g(t) \to 0$ if $t \to \infty$. Then the *inverse entropy problem* is the question whether there is a compact embedding, say, of type (4.139) such that

$$g(t) \sim e(t), \quad 0 \le t < \infty. \tag{4.157}$$

Having (4.128) in mind one may suppose that $g(t) \ge ct^{-K}$ for some $c > 0$, $K > 0$. Similarly one can formulate an *inverse approximation problem* replacing the entropy numbers $\{e_k\}$ in the above comments by the approximation numbers $\{a_k\}$.

Chapter 5
Spaces on smooth domains and manifolds

5.1 Wavelet frames and wavelet-friendly extensions

5.1.1 Introduction

All wavelet expansions in function spaces on domains Ω in $\mathbb{R}^n$ considered so far originate from the u-wavelet basis

$$\big\{\Phi_r^j : j \in \mathbb{N}_0;\ r = 1, \ldots, N_j\big\} \quad \text{with } N_j \in \overline{\mathbb{N}} \tag{5.1}$$

according to Theorem 2.33 and Definitions 2.31, 2.4. The wavelets Φ_r^j have compact supports in Ω,

$$\operatorname{supp} \Phi_r^j \subset \Omega. \tag{5.2}$$

This makes clear that wavelet expansions as considered, for example, in Theorem 3.13 cannot be expected for spaces $A^s_{pq}(\Omega)$ having boundary values at $\Gamma = \partial\Omega$. We discussed this point at the beginning of Section 4.3.2. Nevertheless we obtained in Theorem 2.55 for the Sobolev spaces $W^k_p(\Omega)$ in arbitrary domains and in Theorem 4.23 for all spaces $A^s_{pq}(\Omega)$ in bounded Lipschitz domains so-called constrained wavelet expansions characterising the corresponding spaces in terms of sequence spaces. But these are not wavelet expansions converging in the spaces themselves. We have now a closer look at problems of this type assuming preferably that Ω is a bounded C^∞ domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition 3.4 (iii) or a bounded interval $\Omega = (a, b)$ with $-\infty < a < b < \infty$ if $n = 1$. But at the end of this Chapter 5 we deal briefly with spaces on so-called cellular domains and comment on spaces in Lipschitz domains and (ε, δ)-domains. In Section 6.1 below we return to cellular domains in greater detail.

To avoid any complications in connection with traces we restrict ourselves mostly to the spaces

$$A^s_{pq}(\Omega) \quad \text{with } s > 0,\ 1 \leq p < \infty,\ 1 \leq q < \infty, \tag{5.3}$$

according to Definition 2.1. If $p < 1$ then the trace problem is rather delicate. We return to this point in Section 6.4 below. Otherwise we assume that the reader is familiar with basic facts about traces of, say, Sobolev spaces and classical Besov spaces on the boundary $\Gamma = \partial\Omega$ of a smooth domain Ω. We fix some notation. As usual, Γ is furnished with the $(n-1)$-dimensional Hausdorff measure μ (the standard surface measure). Then $L_p(\Gamma)$ with $1 \leq p < \infty$ is the usual Banach space of all complex-valued μ-measurable functions f on Γ such that

$$\|f \,|L_p(\Gamma)\| = \Big(\int_\Gamma |f(\gamma)|^p \,\mu(\mathrm{d}\gamma)\Big)^{1/p} \tag{5.4}$$

is finite. Since we excluded $p = \infty$ and $q = \infty$ in (5.3) the restriction $S(\Omega) = S(\mathbb{R}^n)|\Omega$ of $S(\mathbb{R}^n)$ on Ω is dense in the space $A^s_{pq}(\Omega)$. If $\varphi \in S(\Omega)$ then the pointwise trace $\varphi(\gamma) = (\mathrm{tr}_\Gamma\, \varphi)(\gamma)$ with $\gamma \in \Gamma$ makes sense. Let $A^s_{pq}(\Omega)$ be as in (5.3). One asks whether there is a positive number c such that

$$\| \mathrm{tr}_\Gamma\, \varphi \,|L_p(\Gamma)\| \le c\, \|\varphi \,|A^s_{pq}(\Omega)\| \quad \text{for all } \varphi \in S(\Omega). \tag{5.5}$$

If this is the case then one defines the *trace operator* tr_Γ,

$$\mathrm{tr}_\Gamma :\ A^s_{pq}(\Omega) \hookrightarrow L_p(\Gamma) \tag{5.6}$$

in the standard way via Cauchy sequences. This is the usual point of view to say what is meant by traces which is sufficient for our (and almost all) purposes. But one can give a more direct (albeit quite often less effective) pointwise definition. It is based on the theory of *Lebesgue points* asking for which $x \in \mathbb{R}^n$ one has

$$g(x) = \lim_{r\to 0} |B(x,r)|^{-1} \int_{B(x,r)} g(y)\,\mathrm{d}y, \tag{5.7}$$

where $B(x,r)$ stands again for a ball centred at $x \in \mathbb{R}^n$ and of radius r and g is locally integrable in $\mathbb{R}^n$. Let g be the distinguished representative of the corresponding equivalence class with (5.7) in all points $x \in \mathbb{R}^n$ for which the right-hand side of (5.7) converges. If one has (5.5) (where one can replace Ω on the right-hand side by $\mathbb{R}^n$) then one obtains for $f \in A^s_{pq}(\Omega)$ and the above distinguished representative $g \in A^s_{pq}(\mathbb{R}^n)$ with $f = g|\Omega$, that

$$(\mathrm{tr}_\Gamma\, f)(\gamma) = f(\gamma) = g(\gamma) = \lim_{r\to 0} |B(\gamma,r)|^{-1} \int_{B(\gamma,r)} g(y)\,\mathrm{d}y \tag{5.8}$$

μ-a.e. (up to a set of μ-measure zero on Γ). But this is a rather sophisticated observation based on the theory of capacity which will not be needed here. Some details can be found in [T01], pp. 260–61, with a reference to [AdH96]. One may also consult in this context [HeN07].

Let $\nu = \nu(\gamma)$ with $\gamma \in \Gamma = \partial\Omega$ be the outer normal at the boundary Γ of the above bounded C^∞ domain Ω in $\mathbb{R}^n$ (bounded interval if $n = 1$). Let $A^s_{pq}(\Omega)$ with $A \in \{B, F\}$ be as in (5.3). If $a \in \mathbb{R}$ then $[a]^-$ is the largest integer strictly less than a. Let

$$\mathrm{tr}^{s,p}_\Gamma :\ f \mapsto \left\{ \mathrm{tr}_\Gamma \tfrac{\partial^j f}{\partial \nu^j} :\ 0 \le j \le \left[s - \tfrac{1}{p}\right]^- \right\} \tag{5.9}$$

(empty if $s \le 1/p$). By the references given later on $\mathrm{tr}^{s,p}_\Gamma$,

$$\mathrm{tr}^{s,p}_\Gamma\, B^s_{pq}(\Omega) = \prod_{j=0}^{[s-1/p]^-} B^{s-\frac{1}{p}-j}_{pq}(\Gamma) \tag{5.10}$$

and

$$\mathrm{tr}^{s,p}_\Gamma\, F^s_{pq}(\Omega) = \prod_{j=0}^{[s-1/p]^-} B^{s-\frac{1}{p}-j}_{pp}(\Gamma) \tag{5.11}$$

are linear and bounded maps onto the indicated (vector) spaces. Let $\tilde{A}^s_{pq}(\Omega)$ be the spaces introduced in Definition 2.1 (ii) with s, p, q and Ω as above. Again by the references and the precise interpretations given in Theorem 5.21 and Remark 5.22 below one has

$$B^s_{pq}(\Omega) = \widetilde{B}^s_{pq}(\Omega) \times \prod_{j=0}^{[s-1/p]^-} B_{pq}^{s-\frac{1}{p}-j}(\Gamma) \tag{5.12}$$

and

$$F^s_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega) \times \prod_{j=0}^{[s-1/p]^-} B_{pp}^{s-\frac{1}{p}-j}(\Gamma) \tag{5.13}$$

if

$$-1 < s - \frac{1}{p} \notin \mathbb{N}_0, \quad 1 \le p, q < \infty. \tag{5.14}$$

If $-1 < s - \frac{1}{p} < 0$ then (5.13), (5.14) means

$$A^s_{pq}(\Omega) = \tilde{A}^s_{pq}(\Omega) \tag{5.15}$$

where again a reference will be given later on. But it is essentially also covered by Proposition 3.19 where one may incorporate $p = 1$ and choose any δ with $\delta < 1$. Although (5.12), (5.13) require a precise interpretation one can imagine that these decompositions pave the way to construct wavelet bases and wavelet frames for the spaces $A^s_{pq}(\Omega)$ under consideration. Wavelet bases for the spaces $\tilde{A}^s_{pq}(\Omega)$ are covered by Theorems 3.13, 3.23 and Corollary 3.25. Next one needs wavelet expansions for the spaces $A^\sigma_{pq}(\Gamma)$ on compact C^∞ manifolds Γ. This will be done in Section 5.1.2. Then one has to transfer the wavelet decomposition for spaces on Γ to Ω which is the subject of Section 5.1.3 where we construct wavelet-friendly extension operators which might be of interest for its own sake. The rest of Section 5.1 deals with the combination of these wavelet expansions resulting in wavelet frames for the spaces in (5.3). In some cases these frames are bases. This is the subject of Sections 5.2 and 5.3. Finally we describe in Section 5.4 some alternative constructions.

5.1.2 Wavelet frames on manifolds

Let Γ be a compact metric space furnished naturally with a topology. Let $n \in \mathbb{N}$. Then Γ is called a *compact n-dimensional C^∞ manifold* if for some $M \in \mathbb{N}$ there is an atlas $\{V_m, \psi_m\}_{m=1}^M$ consisting of open sets V_m in Γ such that $\bigcup_{m=1}^M V_m = \Gamma$ and homeomorphic maps ψ_m,

$$\psi_m : V_m \Longleftrightarrow U_m = \psi_m(V_m) \subset \mathbb{R}^n, \quad m = 1, \ldots, M, \tag{5.16}$$

of V_m onto connected bounded open sets U_m in $\mathbb{R}^n$ such that $\psi_k \circ \psi_m^{-1}$,

$$\psi_k \circ \psi_m^{-1} : \psi_m(V_m \cap V_k) \Longleftrightarrow \psi_k(V_m \cap V_k) \quad \text{if } V_m \cap V_k \neq \emptyset, \tag{5.17}$$

are diffeomorphic C^∞ maps with positive Jacobians. Here (5.17) are the usual compatibility conditions for overlapping open sets V_m. Essentially we are only interested in compact $(n-1)$-dimensional C^∞ manifolds which are boundaries of n-dimensional bounded C^∞ domains according to Definition 3.4 (iii). Then one has natural atlases. This may justify that we do not discuss the above definition on an abstract level. The interested reader may consult [BrL75], Section 1.1, [Hel78], §1, or [Tri86], Section 29.1. We furnish the above n-dimensional compact C^∞ manifold with the n-dimensional Hausdorff measure μ. Details may be found in [Mat95], Section 4. Of course the image measure $\psi_m[\mu]$ restricted to V_m is equivalent to the Lebesgue measure in U_m. As before, $L_p(\Gamma)$ with $0 < p \le \infty$ is the collection of all complex-valued μ-measurable functions on Γ such that

$$\|f\,|L_p(\Gamma)\| = \Big(\int_\Gamma |f(\gamma)|^p\,\mu(\mathrm{d}\gamma)\Big)^{1/p} \tag{5.18}$$

is finite (usual modification if $p=\infty$). Otherwise one can lift via the above mappings ψ_m^{-1} functions, distributions and function spaces from $\mathbb{R}^n$ to Γ. This applies in particular to the space of test functions $D(\Gamma) = C^\infty(\Gamma)$ and its dual $D'(\Gamma)$, the space of distributions on Γ. Let $\{\chi_m\}_{m=1}^M \subset D(\Gamma)$ be a resolution of unity such that

$$\operatorname{supp}\chi_m \subset V_m \quad \text{and} \quad \sum_{m=1}^M \chi_m(\gamma) = 1 \quad \text{if } \gamma\in\Gamma. \tag{5.19}$$

If $f\in D'(\Gamma)$ then

$$(\chi_m f)\circ\psi_m^{-1} \in D'(U_m) \subset S'(\mathbb{R}^n), \qquad m=1,\dots,M. \tag{5.20}$$

Definition 5.1. Let Γ be a compact n-dimensional C^∞ manifold. Let $A\in\{B,F\}$ and let $s\in\mathbb{R}$, $0<p\le\infty$ ($p<\infty$ for the F-spaces) and $0<q\le\infty$. Then

$$A^s_{pq}(\Gamma) = \big\{f\in D'(\Gamma):\ (\chi_m f)\circ\psi_m^{-1}\in A^s_{pq}(\mathbb{R}^n),\ m=1,\dots,M\big\} \tag{5.21}$$

and

$$\|f\,|A^s_{pq}(\Gamma)\| = \sum_{m=1}^M \|(\chi_m f)\circ\psi_m^{-1}\,|A^s_{pq}(\mathbb{R}^n)\|. \tag{5.22}$$

Remark 5.2. It follows by standard arguments (diffeomorphic maps, pointwise multipliers) that these spaces are independent (equivalent quasi-norms) of compatible atlases (local charts) and related resolutions of unity.

We wish to transfer the wavelet expansions for spaces on $\mathbb{R}^n$ according Theorem 1.20 to spaces on Γ. For this purpose one needs counterparts of the wavelets in Theorem 1.20 and of some sequence spaces. It is reasonable to ask for Γ-counterparts of Definitions 2.4, 2.6. The sequence spaces fit in this scheme. But the cancellations of the interior wavelets in (2.33), going back to (1.87)–(1.91), are not preserved if multiplied with C^∞ cut-off functions and the subject of diffeomorphic distortions. The cancellations for the wavelets originate from the corresponding cancellations of the (s,p)-atoms in $\mathbb{R}^n$ according to (1.31). But there is a suitable modification. For this purpose one replaces (1.31), now for all $j \in \mathbb{N}_0$, by

$$\Big| \int_{dQ_{jm}} \psi(x)\, a_{jm}(x)\, \mathrm{d}x \Big| \le D\, 2^{-j(s-\frac{n}{p})-(L+n)j} \sum_{|\alpha|\le L} \sup |D^\alpha \psi(y)|, \tag{5.23}$$

where the supremum is taken over all $y \in dQ_{jm}$. Here D is a positive constant. In other words, we modify Definition 1.5 and the explanations given in Remark 1.6 as follows.

Let s, p, K, L, d be as in Definition 1.5 and let $D > 0$. Then the L_∞-functions $a_{jm}\colon \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called $(s,p)^*$-atoms (more precisely $(s,p)_{K,L,d,D}$-atoms) if one has (1.29), (1.30), and (5.23) for all functions ψ having classical derivatives up to order L in dQ_{jm}.

In other words, one replaces (1.31) by (5.23) for given $D > 0$ and all ψ. Expanding ψ in its Taylor polynomial with remainder term of order L and off-point $2^{-j}m$ then (5.23) follows from (1.31) for all

$$D \ge D(n,L), \quad n \in \mathbb{N},\; L \in \mathbb{N}_0, \tag{5.24}$$

where $D(n,L)$ is some positive constant depending only on n and L (which can be calculated explicitly). Then any $(s,p)_{K,L,d}$-atom according to Definition 1.5 and Remark 1.6 is an $(s,p)_{K,L,d,D}$-atom if D satisfies (5.24).

Proposition 5.3. *Theorem 1.7 remains valid if one replaces (s,p)-atoms (more precisely $(s,p)_{K,L,d}$-atoms) by $(s,p)^*$-atoms (more precisely $(s,p)_{K,L,d,D}$-atoms with $D \ge D(n,L)$ according to (5.24)).*

Remark 5.4. One has to show that (1.34), (1.35) and (1.34), (1.37) remain valid for the larger class of $(s,p)^*$-atoms according to the above definitions. This has been done in detail in [Skr98b]. Multiplications with smooth functions and (local) diffeomorphic maps of $(s,p)^*$-atoms preserve essentially the crucial assumptions (1.29), (1.30), (5.23).

After these preparations one can now introduce the Γ-counterpart of Definitions 2.4 and 2.6. Let Γ be again a compact n-dimensional C^∞ manifold. Let $B(\gamma,\varrho)$ be a ball centred at $\gamma \in \Gamma$ and of radius $\varrho > 0$ (with respect to the metric on Γ). For some $c > 0$ let

$$\mathbb{Z}_\Gamma = \big\{ \gamma^j_r \in \Gamma : \; j \in \mathbb{N}_0;\; r = 1,\ldots,N_j \big\}, \quad N_j \in \mathbb{N}, \tag{5.25}$$

with (typically) $N_j \sim 2^{jn}$ such that

$$\gamma^j_{r'} \notin B\big(\gamma^j_r, c2^{-j}\big), \quad j \in \mathbb{N}_0,\; r \ne r'. \tag{5.26}$$

This is the counterpart of (2.24), (2.25). Of interest later on are sets $\mathbb{Z}_\Gamma$ with

$$\Gamma = \bigcup_{r=1}^{N_j} B\big(\gamma^j_r, c'\, 2^{-j}\big), \quad j \in \mathbb{N}_0, \tag{5.27}$$

for some $c' > 0$. Let $C^u(\Gamma)$ be the usual space of complex-valued functions having classical derivatives up to order $u \in \mathbb{N}$ inclusively. Since Γ is compact there is no need to bother about invariantly defined derivatives. We simply write $D^\alpha f$ for $f \in C^u(\Gamma)$ and

$$\|f\,|C^u(\Gamma)\| \sim \sum_{|\alpha| \le u} \sup_{\gamma \in \Gamma} |D^\alpha f(\gamma)| \tag{5.28}$$

in the interpretation via local charts. Recall that μ in (5.18) is the n-dimensional Hausdorff measure on Γ.

Definition 5.5. Let Γ be a compact n-dimensional C^∞ manifold and let $\mathbb{Z}_\Gamma$ be as in (5.25), (5.26). Let $n \in \mathbb{N}$.

(i) Then the collection

$$\Phi^\Gamma = \left\{\Phi^j_r : \ j \in \mathbb{N}_0; \ r = 1, \ldots, N_j\right\} \subset C^u(\Gamma) \tag{5.29}$$

of real functions is called a u-wavelet system if for some positive numbers c_1, c_2,

$$\operatorname{supp} \Phi^j_r \subset B\left(\gamma^j_r, c_1 2^{-j}\right), \quad j \in \mathbb{N}_0; \ r = 1, \ldots, N_j, \tag{5.30}$$

and for all $\alpha \in \mathbb{N}_0$ with $|\alpha| \le u$,

$$|D^\alpha \Phi^j_r(\gamma)| \le c_2\, 2^{j\frac{n}{2} + j|\alpha|}, \quad j \in \mathbb{N}_0; \ r = 1, \ldots, N_j. \tag{5.31}$$

(ii) The above u-wavelet system is called oscillating if, in addition, for some $c_3 > 0$,

$$\left| \int_\Gamma \psi(\gamma)\, \Phi^j_r(\gamma)\, \mu(\mathrm{d}\gamma) \right| \le c_3\, 2^{-j\frac{n}{2} - ju}\, \|\psi\,|C^u(\Gamma)\|, \tag{5.32}$$

$j \in \mathbb{N}_0; r = 1, \ldots, N_j$.

Remark 5.6. This is the counterpart of Definition 2.4 based on (2.32), (2.33), (2.27). As there we prefer for wavelets (in contrast to atoms) an L_2-normalisation. This explains the factors $2^{jn/2}$ and $2^{-jn/2} = 2^{jn/2}{\cdot}2^{-jn}$ in (5.31), (5.32). Furthermore we rely on the same coupling of smoothness and cancellation as in (1.87)–(1.91) expressed by $u \in \mathbb{N}$. The above wavelets are more qualitative than there counterparts in Definition 2.4 and there is no need to distinguish any longer between basic wavelets as in (2.32) and interior wavelets as in (2.33). It is obviously immaterial whether one requires (5.32) for $j \in \mathbb{N}$ or $j \in \mathbb{N}_0$ (including the starting wavelets). In terms of local charts, (5.32) reduces to (5.23) with $L = u$ (having in mind the different normalisations).

Definition 5.7. Let Γ be a compact n-dimensional C^∞ manifold and let $\mathbb{Z}_\Gamma$ be as in (5.25), (5.26). Let χ_{jr} be the characteristic functions for the balls in (5.26). Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then $b^s_{pq}(\mathbb{Z}_\Gamma)$ is the collection of all sequences

$$\lambda = \left\{\lambda^j_r \in \mathbb{C} : \ j \in \mathbb{N}_0; \ r = 1, \ldots, N_j\right\}, \quad N_j \in \mathbb{N}, \tag{5.33}$$

such that

$$\|\lambda \,|b^s_{pq}(\mathbb{Z}_\Gamma)\| = \Big(\sum_{j=0}^\infty 2^{j(s-\frac{n}{p})q} \Big(\sum_{r=1}^{N_j} |\lambda^j_r|^p \Big)^{q/p} \Big)^{1/q} < \infty, \tag{5.34}$$

and $f^s_{pq}(\mathbb{Z}_\Gamma)$ is the collection of all sequences (5.33) such that

$$\|\lambda \,| f^s_{pq}(\mathbb{Z}_\Gamma)\| = \Big\| \Big(\sum_{j=0}^\infty \sum_{r=1}^{N_j} 2^{jsq} |\lambda^j_r \, \chi_{jr}(\cdot)|^q \Big)^{1/q} |L_p(\Gamma) \Big\| < \infty \tag{5.35}$$

(obviously modified if $p = \infty$ and/or $q = \infty$).

Remark 5.8. This is the direct counterpart of Definition 2.6 where the quasi-norm in $L_p(\Gamma)$ is given by (5.18) independently of local charts. As mentioned there if ψ_m in (5.16) is applied to μ (image measure) then $\psi_m[\mu]$ is equivalent to the Lebesgue measure in U_m. Then one has the same situation as in Remark 2.7 making clear that $f^s_{pq}(\mathbb{Z}_\Gamma)$ is independent of the admitted characteristic functions χ_{jr} and admitted atlases. The Hilbert space $L_2(\Gamma)$ is furnished with the scalar product

$$(f, g)_\Gamma = \int_\Gamma f(\gamma)\, \overline{g(\gamma)}\, \mu(\mathrm{d}\gamma). \tag{5.36}$$

The corresponding scalar product with V_m in place of Γ can be transferred by (5.16) to U_m,

$$(f, g)_{V_m} = \int_{U_m} \big(f \circ \psi_m^{-1}\big)(x) \cdot \overline{\big(g \circ \psi_m^{-1}\big)(x)} \cdot \frac{\mathrm{d}\mu}{\mathrm{d}x}(x)\, \mathrm{d}x. \tag{5.37}$$

One may assume that the Radon–Nikodym derivative $\frac{d\mu}{dx}$ is a C^∞ function in U_m with $\frac{\mathrm{d}\mu}{\mathrm{d}x}(x) \ge c > 0$ if $x \in U_m$. This can be justified by transferring the Lebesgue measure in U_m, multiplied with a smooth resolution of unity, to Γ. Then one obtains a measure with the desired property which is at least equivalent to the above Hausdorff measure μ. We identify μ with this measure. (Recall that in what follows we are only interested in C^∞ manifolds which are boundaries of bounded C^∞ domains according to Definition 3.4 (iii) where the above assertion can be justified directly).

After these preparations one obtains now the following counterparts of Theorem 1.20 for spaces on $\mathbb{R}^n$ and of Theorem 1.37 for spaces on $\mathbb{T}^n$. The numbers σ_p and σ_{pq} have the same meaning as in (4.15). We use $\sim$ as in (3.4), (3.5).

Theorem 5.9. *Let Γ be a compact n-dimensional C^∞ manifold. Let $A^s_{pq}(\Gamma)$ be the spaces as introduced in Definition 5.1. For any $u \in \mathbb{N}$ there are two oscillating u-wavelet systems $\{\Phi^j_r\}$ and $\{\Psi^j_r\}$ according to Definition 5.5* (ii) *with the following properties.*

(i) *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, and*

$$u > \max(s, \sigma_{pq} - s). \tag{5.38}$$

Then $f \in D'(\Gamma)$ *is an element of* $F^s_{pq}(\Gamma)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r \, 2^{-jn/2} \, \Phi^j_r, \quad \lambda \in f^s_{pq}(\mathbb{Z}_\Gamma), \tag{5.39}$$

unconditional convergence being in $F^s_{pq}(\Gamma)$ *if* $q < \infty$ *and in* $F^{s-\varepsilon}_{pq}(\Gamma)$ *for any* $\varepsilon > 0$ *if* $q = \infty$*. Furthermore,*

$$\| f \,| F^s_{pq}(\Gamma)\| \sim \inf \|\lambda \,| f^s_{pq}(\mathbb{Z}_\Gamma)\| \tag{5.40}$$

where the infimum is taken over all admissible representations (5.39)*. Any* $f \in F^s_{pq}(\Gamma)$ *admits the distinguished representation* (5.39)*,*

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r(f) \, 2^{-jn/2} \, \Phi^j_r, \quad \lambda^j_r(f) = 2^{jn/2} \big(f, \Psi^j_r \big)_\Gamma, \tag{5.41}$$

with

$$\| f \,| F^s_{pq}(\Gamma)\| \sim \|\lambda(f) \,| f^s_{pq}(\mathbb{Z}_\Gamma)\| \tag{5.42}$$

(u*-wavelet frame*).

(ii) *Let* $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$ *and*

$$u > \max(s, \sigma_p - s). \tag{5.43}$$

Then $f \in D'(\Gamma)$ *is an element of* $B^s_{pq}(\Gamma)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r \, 2^{-jn/2} \, \Phi^j_r, \quad \lambda \in b^s_{pq}(\mathbb{Z}_\Gamma), \tag{5.44}$$

unconditional convergence being in $B^s_{pq}(\Gamma)$ *if* $p < \infty$, $q < \infty$ *and in* $B^{s-\varepsilon}_{pq}(\Gamma)$ *for any* $\varepsilon > 0$ *if* $p = \infty$ *and/or* $q = \infty$*. Furthermore,*

$$\| f \,| B^s_{pq}(\Gamma)\| \sim \inf \|\lambda \,| b^s_{pq}(\mathbb{Z}_\Gamma)\| \tag{5.45}$$

where the infimum is taken over all admissible representations (5.44)*. Any* $f \in B^s_{pq}(\Gamma)$ *admits the distinguished representation* (5.41), (5.44)*, with*

$$\| f \,| B^s_{pq}(\Gamma)\| \sim \|\lambda(f) \,| b^s_{pq}(\mathbb{Z}_\Gamma)\| \tag{5.46}$$

(u*-wavelet frame*).

Proof. *Step* 1. Let $A \in \{B, F\}$ and correspondingly $a \in \{b, f\}$. Let $f \in D'(\Gamma)$ be given by (5.39), (5.44) for some u-wavelet system $\big\{\Phi^j_r\big\}$ with (5.38), (5.43). We rely on (5.21). One has

$$(\chi_m f) \circ \psi_m^{-1} = \sum_{j,r} \lambda^j_r \, 2^{-jn/2} \, \big(\chi_m \, \Phi^j_r\big) \circ \psi_m^{-1}. \tag{5.47}$$

Adapting the normalising factors one has by Definition 5.5 that (5.47) is an expansion in terms of $(s,p)^*$-atoms to which Proposition 5.3 can be applied. Then one obtains that $(\chi_m f) \circ \psi_m^{-1} \in A^s_{pq}(\mathbb{R}^n)$ and

$$\| f \,| A^s_{pq}(\Gamma)\| \le c \, \|\lambda \,| a^s_{pq}(\mathbb{Z}_\Gamma)\|. \tag{5.48}$$

Step 2. As for the converse one expands

$$(\chi_m f) \circ \psi_m^{-1} = \sum_{j,G,l} \lambda^{j,G}_l \big(\chi_m f \circ \psi_m^{-1}\big)\, 2^{-jn/2}\, \Psi^j_{G,l} \tag{5.49}$$

according to Theorem 1.20 with respect to the wavelet basis $\{\Psi^j_{G,l}\}$, where

$$\lambda^{j,G}_l \big(\chi_m f \circ \psi_m^{-1}\big) = 2^{jn/2} \int_{\mathbb{R}^n} (\chi_m f \circ \psi_m^{-1})(x)\, \Psi^j_{G,l}(x)\, \mathrm{d}x. \tag{5.50}$$

One may assume that $\operatorname{supp} \Psi^j_{G,l} \subset U_m$ according to (5.16) for all $\Psi^j_{G,l}$ with

$$\operatorname{supp} \Psi^j_{G,l} \cap \operatorname{supp} \big(\chi_m \circ \psi_m^{-1}\big) \neq \emptyset. \tag{5.51}$$

We transfer (5.49) from U_m to Γ. Then (5.37) and summation over m give (5.41). Using (5.22) one obtains (5.42), (5.45). □

Remark 5.10. We proved a little bit more than stated. One has (5.48) for any expansion by appropriately re-normalised $(s,p)^*$-atoms according to Proposition 5.3 and transferred from $\mathbb{R}^n$ to Γ. In other words there is a full counterpart of Theorem 1.7 in the version of Proposition 5.3 for spaces $A^s_{pq}(\Gamma)$ on compact n-dimensional C^∞ manifolds Γ.

Remark 5.11. In (1.100), (1.101) we recalled what is meant by a (unconditional) basis in quasi-Banach spaces. The notation of a *frame* in quasi-Banach spaces is not so fixed but in common use nowadays. We explain our understanding of this notation taking $B^s_{pq}(\Gamma)$ as a proto-type. A countable system $\{\Phi^j_r\}$ is called a (stable) *frame* in $B^s_{pq}(\Gamma)$ if it has the following two properties.

1. (*Stability*) There is a natural sequence space, $b^s_{pq}(\mathbb{Z}_\Gamma)$, such that any $f \in B^s_{pq}(\Gamma)$ can be represented by (5.44) with (5.45).
2. (*Optimality*) There are linear and continuous functionals

$$\lambda^j_r \in B^s_{pq}(\Gamma)' \colon f \in B^s_{pq}(\Gamma) \mapsto \lambda^j_r(f) \in \mathbb{C}, \tag{5.52}$$

such that any $f \in B^s_{pq}(\Gamma)$ can be represented by

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r(f)\, 2^{-jn/2}\, \Phi^j_r, \quad \lambda(f) \in b^s_{pq}(\mathbb{Z}_\Gamma), \tag{5.53}$$

with (5.46).

To avoid any misunderstanding we recall that (5.46) means that there are two numbers $0 < C_1 \le C_2 < \infty$ such that

$$C_1 \, \| f \,|B^s_{pq}(\Gamma)\| \le \|\lambda(f)\,|b^s_{pq}(\mathbb{Z}_\Gamma)\| \le C_2 \| f \,|B^s_{pq}(\Gamma)\| \tag{5.54}$$

for all $f \in B^s_{pq}(\Gamma)$. If the underlying space is a separable Hilbert space, then ℓ_2 is the natural related sequence space. If one starts with a Banach space or even quasi-Banach space then the search for a suitable sequence space is part of the task. Unconditional convergence of (5.44) or (5.53) can only be expected if the underlying space is separable, this means $p < \infty$, $q < \infty$ in case of $B^s_{pq}(\Gamma)$. If the given space is not separable, for example $B^s_{pq}(\Gamma)$ with $\max(p,q) = \infty$, then convergence takes place in a larger space, for example $B^{s-\varepsilon}_{pq}(\Gamma)$ with $\varepsilon > 0$, but the outcome belongs to the given space. Furthermore (5.52), (5.53) and suitable counterparts for other quasi-Banach spaces require that the space considered has a sufficiently rich dual space. But this is not always the case. Since, for example, $L_p(\mathbb{R}^n)' = \{0\}$ if $0 < p < 1$, nothing like (5.53), (5.52) can be expected. But this unpleasant effect happens also for some other more interesting spaces. We return to this point in Section 6.2.

Remark 5.12. One may ask whether there are u-wavelet systems according to Definition 5.5 which are not only frames but even (unconditional) bases. In general this is not so clear. But it is possible in some cases and for some modified u-wavelet systems. This will be the subject of Sections 5.2, 5.3.

Remark 5.13. As described in Introduction 5.1.1 it is the main topic of Chapter 5 to find wavelet frames and wavelet bases for spaces $A^s_{pq}(\Omega)$ using the decompositions (5.12), (5.13). For this purpose we need the above considerations applied to the compact $(n-1)$-dimensional C^∞ manifold $\Gamma = \partial\Omega$. But it would be desirable to study wavelets (frames and bases) on more general C^∞ manifolds Γ. In [T92], Chapter 7, and the underlying papers we introduced and studied spaces $B^s_{pq}(\Gamma)$ and $F^s_{pq}(\Gamma)$ on (non-compact) n-dimensional Riemannian manifolds with positive injectivity radius and bounded geometry and on Lie groups. The further development of this theory is mostly due to L. Skrzypczak, [Skr97], [Skr98a], [Skr98b], [Skr03]. In these papers one finds in particular an elaborated theory of atomic decompositions based on atoms of the same type as used in Proposition 5.3, where conditions of type (5.23) play a decisive role. Wavelet frames in these spaces have been studied in the recent paper [Skr08].

5.1.3 Wavelet-friendly extensions

In Introduction 5.1.1 we outlined our method in this chapter. We reduce the question of wavelet frames and wavelet bases for the spaces $A^s_{pq}(\Omega)$ in (5.3) to corresponding assertions for the spaces in the decompositions (5.12), (5.13). By Theorem 3.13 we have interior wavelet bases for all spaces $\tilde{A}^s_{pq}(\Omega)$ of interest. As for the spaces $B^\sigma_{pq}(\Gamma)$ on the boundary $\Gamma = \partial\Omega$ we wish to use the wavelet frames (or related wavelet bases discussed later on) according to Theorem 5.9. One has to shift the wavelets on the compact $(n-1)$-dimensional C^∞ manifold Γ to the n-dimensional bounded C^∞ domain Ω creating wavelets in $\overline{\Omega}$ having all the expected properties of wavelets in $\overline{\Omega}$, but no compact supports in Ω. This will be done by constructing wavelet-friendly extension operators which might be of interest for its own sake. For this reasons we prove the corresponding assertion in full generality. First we adapt some formulations from Introduction 5.1.1.

If $p \geq 1$ then one can replace $L_p(\Gamma)$ in (5.5), (5.6) by $L_1(\Gamma)$. Hence we ask for a positive constant c such that

$$\| \operatorname{tr}_\Gamma \varphi \,|L_1(\Gamma)\| \leq c\, \|\varphi\,|A^s_{pq}(\Omega)\| \quad \text{for all } \varphi \in S(\Omega). \tag{5.55}$$

This does not change to say what is meant by traces for the spaces in (5.3) and also not the discussion in (5.7), (5.8) which is essentially an L_1-matter. But we extend now (5.55) to spaces $A^s_{pq}(\Omega)$ with $p = \infty$ and/or $q = \infty$ and to spaces with $p < 1$. For given $r \in \mathbb{N}_0$ and $u \in \mathbb{N}$ with $r < u$, we will ask for common trace operators (from Ω to Γ), now denoted by $\operatorname{tr}^r_\Gamma$, and for common extension operators (from Γ to Ω), now denoted by $\operatorname{ext}^{r,u}_\Gamma$, in natural (s,p,q)-regions

$$0 < p \leq \infty, \quad 0 < q \leq \infty, \quad r + \frac{1}{p} + \sigma^{n-1}_p < s < u, \tag{5.56}$$

where

$$\sigma^{n-1}_p = (n-1)\Big(\frac{1}{p} - 1\Big)_+, \quad 0 < p \leq \infty,\ 2 \leq n \in \mathbb{N}, \tag{5.57}$$

is the $(n-1)$-dimensional version of σ_p in (4.15) indicating the dimension $n-1$ to avoid any misunderstanding. Furthermore, if $\{B_l : l = 1,\ldots,L\}$ are (complex) quasi-Banach spaces then the collection of all $b = \{b_1,\ldots,b_L\}$ with $b_l \in B_l$ and quasi-normed by

$$\sum_{l=1}^{L} \|b_l\,|B_l\| \quad \text{is denoted as} \quad \prod_{l=1}^{L} B_l. \tag{5.58}$$

Instead of $\operatorname{tr}^{s,p}_\Gamma$ in (5.9) we write now

$$\operatorname{tr}^r_\Gamma : f \mapsto \big\{ \operatorname{tr}_\Gamma \tfrac{\partial^j f}{\partial \nu^j} : 0 \leq j \leq r \big\}. \tag{5.59}$$

Then we obtain by [T83], Theorem 3.3.3, p. 200, that

$$\operatorname{tr}^r_\Gamma B^s_{pq}(\Omega) = \prod_{j=0}^{r} B^{s-\frac{1}{p}-j}_{pq}(\Gamma) \tag{5.60}$$

if

$$0 < p \leq \infty, \quad 0 < q \leq \infty, \quad r + \frac{1}{p} + \sigma^{n-1}_p < s, \tag{5.61}$$

and

$$\operatorname{tr}^r_\Gamma F^s_{pq}(\Omega) = \prod_{j=0}^{r} B^{s-\frac{1}{p}-j}_{pp}(\Gamma) \tag{5.62}$$

if

$$0 < p < \infty, \quad 0 < q \le \infty, \quad r + \frac{1}{p} + \sigma^{n-1}_p < s. \tag{5.63}$$

We add a comment how (5.55) is related to these trace assertions. First we remark that

$$L_p(\Gamma) \hookrightarrow L_1(\Gamma) \quad \text{if } 1 \le p \le \infty. \tag{5.64}$$

By (5.60)–(5.63) it follows that it is immaterial to replace (5.5) by (5.55) if $p \ge 1$. If $0 < p < 1$ then one has for $\delta > 0$,

$$s = \delta + \frac{1}{p} + (n-1)\Big(\frac{1}{p} - 1\Big) = \frac{n}{p} - n + 1 + \delta. \tag{5.65}$$

Then it follows from well-known embedding theorems as one may found in [T83], Theorem 3.3.1, p. 196, that

$$B^s_{pq}(\Omega) \hookrightarrow B^{1+\delta}_{1,q}(\Omega) \hookrightarrow L_1(\Gamma). \tag{5.66}$$

Again (5.55) makes sense. If $p < \infty$, $q < \infty$, then $S(\Omega)$ is dense in $A^s_{pq}(\Omega)$ and, as said, tr_Γ, and then also $\operatorname{tr}^r_\Gamma$, are defined by completion. If $p = \infty$ then $B^s_{\infty,q}(\Omega)$ with $s > 0$ is embedded in the space of continuous functions and tr_Γ makes sense pointwise. If $q = \infty$ then one has

$$B^s_{p,\infty}(\Omega) \hookrightarrow B^{s-\varepsilon}_{p,1}(\Omega) \quad \text{for any } \varepsilon > 0. \tag{5.67}$$

Let $\varepsilon > 0$ be small such that one has (5.61) with $s-\varepsilon$ in place of s. Now one can define tr_Γ for $B^s_{p,\infty}(\Omega)$ by restriction of tr_Γ for $B^{s-\varepsilon}_{p,1}(\Omega)$ to $B^s_{p,\infty}(\Omega)$. Hence (5.55) is always meaningful. As far as (5.60)–(5.63) is concerned we have nothing new to add. The assertions in [T83], Theorem 3.3.3, p. 200, cover also the existence of corresponding extension operators. But this is based on Fourier-analytical arguments and of little use for our purpose. In [T92], Section 4.4, we returned to the trace problem, based on atoms. Although nearer to our recent intentions it does not cover our needs. For this reason we construct explicitly a *wavelet-friendly extension operator* $\operatorname{ext}^{r,u}_\Gamma$, where $r \in \mathbb{N}_0$ and $u \in \mathbb{N}$ have the same meaning as in (5.56), (5.61).

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ according to Definition 3.4 (iii) with $2 \le n \in \mathbb{N}$ and let $\Gamma = \partial\Omega$ be its $(n-1)$-dimensional boundary considered as an $(n-1)$-dimensional compact C^∞ manifold. We recalled at the beginning of Section 5.1.2 what is meant by compact C^∞ manifolds (without boundaries). We need some preparations. Let ν be the outer normal at Γ and let

$$d(x) \sim \begin{cases} \operatorname{dist}(x,\Gamma) & \text{if } x \in \Omega, \\ -\operatorname{dist}(x,\Gamma) & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases} \tag{5.68}$$

be a C^∞ function in a tubular neighbourhood

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Gamma) < \varepsilon\} \tag{5.69}$$

of Γ for some (small) $\varepsilon > 0$. We may assume that

$$\frac{\partial^k}{\partial \nu^k} d^l(x) = 0 \text{ if } l \neq k \quad \text{and} \quad \frac{\partial^k}{\partial \nu^k} d^k(x) = k!. \tag{5.70}$$

Then Γ_ε can be furnished for small $\varepsilon > 0$ with curvilinear coordinates

$$\Gamma_\varepsilon \ni x = \gamma = (\gamma', \gamma_n) \quad \text{with } \gamma' \in \Gamma \text{ and } \gamma_n = d(\gamma) \tag{5.71}$$

pointing in the ν-direction in agreement with (5.70). Let $\chi(t)$ be a real function on $\mathbb{R}$ with

$$\chi \in D(\mathbb{R}), \quad \operatorname{supp} \chi \subset (-\varepsilon, \varepsilon), \quad \chi(t) = 1 \text{ if } |t| \le \varepsilon/2. \tag{5.72}$$

For given $L \in \mathbb{N}_0$ one may assume in addition that

$$\int_{\mathbb{R}} \chi(t)\, t^l \,\mathrm{d}t = 0, \quad l = 0, \dots, L-1, \tag{5.73}$$

(no condition if $L = 0$). This is of some service in the following theorem for spaces $F^s_{pq}(\Omega)$ with $q > 0$ small. It will not be indicated. Let $\{\Phi^j_l\}$ and $\{\Psi^j_l\}$ be two real u-wavelet systems according to Definition 5.5 (i) on the compact $(n-1)$-dimensional manifold Γ. Let

$$\lambda^j_l(h) = 2^{j\frac{n-1}{2}} \left(h, \Psi^j_l\right)_\Gamma = 2^{j\frac{n-1}{2}} \int_\Gamma h(\gamma')\, \Psi^j_l(\gamma')\, \mu(\mathrm{d}\gamma') \tag{5.74}$$

where $(\cdot,\cdot)_\Gamma$ is the scalar product (5.36) on Γ (recall that Ψ^j_l is real). Then we put for some $r \in \mathbb{N}_0$ and $\{g_k\}^r_{k=0} \subset L_1(\Gamma)$,

$$\begin{aligned} g &= \operatorname{Ext}^{r,u}_\Gamma \{g_0, \dots, g_r\} \\ &= \sum_{k=0}^r \sum_{j=0}^\infty \sum_{l=1}^{N_j} \frac{1}{k!} \lambda^j_l(g_k)\, 2^{-j\frac{n-1}{2}}\, \gamma_n^k\, \chi(2^j \gamma_n)\, \Phi^j_l(\gamma') \\ &= \sum_{k=0}^r \sum_{j=0}^\infty \sum_{l=1}^{N_j} \frac{1}{k!} \lambda^j_l(g_k)\, 2^{-jk}\, 2^{-jn/2}\, \Phi^{j,k}_l(\gamma) \end{aligned} \tag{5.75}$$

with $\lambda^j_l(g_k)$ as in (5.74) and

$$\Phi^{j,k}_l(\gamma) = 2^{jk}\, \gamma_n^k\, \chi(2^j\gamma_n)\, 2^{j/2}\, \Phi^j_l(\gamma'). \tag{5.76}$$

Each term in (5.75) makes sense. Under the restrictions for g_k in the theorem below it follows that (5.75) is an atomic decomposition in the related spaces $A^s_{pq}(\mathbb{R}^n)$. This ensures the convergence of (5.75) in the same way as in Theorem 1.7, Remark 1.8 and the references given there. We will not stress this point in the sequel. In Remark 4.2 we said what is meant by a *common extension operator*. This will be used, obviously adapted. Let $\mathrm{re}_\Omega = \mathrm{re}$ be the restriction operator according to (4.1).

Theorem 5.14. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition 3.4* (iii) *or a bounded interval in $\mathbb{R}$ if $n = 1$. Let $A^s_{pq}(\Omega)$ and $A^s_{pq}(\Gamma)$ be the spaces as introduced in Definition 2.1* (i) *on Ω and in Definition 5.1 on the compact $(n-1)$-dimensional manifold $\Gamma = \partial\Omega$ (the two endpoints of the interval if $n = 1$). Let $r \in \mathbb{N}_0$, $u \in \mathbb{N}$ with $r < u$ and $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon \leq 1$. Let tr^r_Γ be the trace operator* (5.59) *with* (5.60), (5.61) *and* (5.62), (5.63). *Let $\mathrm{Ext}^{r,u}_\Gamma$ be given by* (5.75) *with* (5.74), (5.76) *where $\{\Phi^j_l\}$ and $\{\Psi^j_l\}$ are the two u-wavelet systems on Γ from Theorem 5.9 and $L \geq n\big(\frac{1}{\varepsilon} - 1\big)$ in* (5.73). *Then*

$$\mathrm{ext}^{r,u}_\Gamma = \mathrm{re}_\Omega \circ \mathrm{Ext}^{r,u}_\Gamma$$

with

$$\mathrm{Ext}^{r,u}_\Gamma : \{g_0, \ldots, g_r\} \mapsto g, \tag{5.77}$$

according to (5.75) *is a common extension operator for*

$$\begin{gathered} \mathrm{ext}^{r,u}_\Gamma : \prod_{k=0}^r B^{s-\frac{1}{p}-k}_{pq}(\Gamma) \hookrightarrow B^s_{pq}(\Omega), \\ 0 < p \leq \infty, \quad 0 < q \leq \infty, \quad r + \tfrac{1}{p} + \sigma^{n-1}_p < s < u, \end{gathered} \tag{5.78}$$

and

$$\begin{gathered} \mathrm{ext}^{r,u}_\Gamma : \prod_{k=0}^r B^{s-\frac{1}{p}-k}_{pp}(\Gamma) \hookrightarrow F^s_{pq}(\Omega), \\ 0 < p < \infty, \quad \varepsilon \leq q \leq \infty, \quad r + \tfrac{1}{p} + \sigma^{n-1}_p < s < u, \end{gathered} \tag{5.79}$$

with

$$\mathrm{tr}^r_\Gamma \circ \mathrm{ext}^{r,u}_\Gamma = \mathrm{id}, \quad \textit{identity in } \prod_{k=0}^r B^{s-\frac{1}{p}-k}_{pq}(\Gamma). \tag{5.80}$$

Proof. *Step* 1. We prove the theorem for the B-spaces. For p, q, s as in (5.78) we need now only the weaker version

$$\mathrm{tr}^r_\Gamma : B^s_{pq}(\Omega) \hookrightarrow \prod_{k=0}^r B^{s-\frac{1}{p}-k}_{pq}(\Gamma) \tag{5.81}$$

of (5.60) based on [T83], Theorem 3.3.3, p. 200. Let for the same p, q, s,

$$g_k \in B^{s-\frac{1}{p}-k}_{pq}(\Gamma), \quad k = 0, \ldots, r. \tag{5.82}$$

Since $u > s - \frac{1}{p} - k > \sigma^{n-1}_p$ for $k = 0, \ldots, r$ one can apply Theorem 5.9 to all spaces on the right-hand side of (5.82) based on common u-wavelet systems $\{\Phi^j_l\}$ and $\{\Psi^j_l\}$. One has by this assertion and (5.34) that

$$g_k(\gamma') = \sum_{j=0}^\infty \sum_{l=1}^{N_j} \lambda^j_l(g_k)\, 2^{-j\frac{n-1}{2}}\, \Phi^j_l(\gamma'), \quad \gamma' \in \Gamma, \tag{5.83}$$

with

$$\|g_k \,|B_{pq}^{s-\frac{1}{p}-k}(\Gamma)\| \sim \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p}-k)q} \Big(\sum_{l=1}^{N_j} |\lambda_l^j(g_k)|^p \Big)^{q/p} \Big)^{1/q} \tag{5.84}$$

(usual modification if $p = \infty$ and/or $q = \infty$),

$$\lambda_l^j(g_k) = 2^{j\frac{n-1}{2}} \int_\Gamma g_k(\gamma')\, \Psi_l^j(\gamma')\, \mu(\mathrm{d}\gamma'), \tag{5.85}$$

having in mind that Γ is $(n-1)$-dimensional. By (5.30), (5.31) (with γ' in place of γ) one obtains for $\Phi_l^{j,k}$ in (5.76) that

$$\operatorname{supp} \Phi_l^{j,k} \subset Q_{jl} = B\big(\gamma_l^{\prime j}, c_1\, 2^{-j}\big) \times \big[-\varepsilon 2^{-j}, \varepsilon 2^{-j}\big] \tag{5.86}$$

centred at $\gamma_l^j = (\gamma_l^{\prime j}, 0)$ with diameter $\sim 2^{-j}$ and

$$\big|D^\alpha \Phi_l^{j,k}(\gamma)\big| \le c\, 2^{j\frac{n}{2} + j|\alpha|}, \qquad j \in \mathbb{N}_0;\ l = 1, \ldots, N_j, \tag{5.87}$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le u$ (in $\mathbb{R}^n$). With

$$a_{kl}^j = 2^{-jn/2}\, \Phi_l^{j,k} \quad \text{and} \quad \mu_{kl}^j = \frac{1}{k!}\, \lambda_l^j(g_k)\, 2^{-jk} \tag{5.88}$$

one can re-write (5.75) as

$$g = \sum_{k,j,l} \mu_{kl}^j\, a_{kl}^j \tag{5.89}$$

where a_{kl}^j are L_∞-normalised atoms in $B_{pq}^s(\mathbb{R}^n)$ (no moment conditions are required). Here one needs that

$$n\Big(\frac{1}{p} - 1\Big)_+ = \sigma_p^n = \sigma_p < s < u, \tag{5.90}$$

covered by (5.78) and (5.65). Then it follows from Theorem 1.7 and (5.84) that

$$\begin{aligned}
\|g \,|B_{pq}^s(\mathbb{R}^n)\| &\le c \sum_{k=0}^{r} \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \Big(\sum_{l=1}^{N_j} |\mu_{kl}^j|^p \Big)^{q/p} \Big)^{1/q} \\
&\le c' \sum_{k=0}^{r} \Big(\sum_{j=0}^{\infty} 2^{j(s-k-\frac{n}{p})q} \Big(\sum_{l=1}^{N_j} |\lambda_l^j(g_k)|^p \Big)^{q/p} \Big)^{1/q} \\
&\le c'' \sum_{k=0}^{r} \|g_k \,|B_{pq}^{s-\frac{1}{p}-k}(\Gamma)\|.
\end{aligned} \tag{5.91}$$

In particular, $g \in B_{pq}^s(\mathbb{R}^n)$. The coefficients in (5.89), (5.88) depend linearly on g_k. Hence $\mathrm{Ext}_\Gamma^{r,u}$ is (5.77), (5.75) is a linear and bounded map,

$$\mathrm{Ext}_\Gamma^{r,u}:\ \prod_{k=0}^{r} B_{pq}^{s-\frac{1}{p}-k}(\Gamma) \hookrightarrow B_{pq}^s(\mathbb{R}^n). \tag{5.92}$$

We check that $\mathrm{Ext}^{r,u}_\Gamma$ is an extension operator. Taking the derivative with respect to the normal ν then one obtains by (5.70)–(5.72), (5.75) and (5.83) that

$$\begin{aligned}\left(\frac{\partial^k g}{\partial \nu^k}\right)(\gamma',0) &= \sum_{j=0}^{\infty}\sum_{l=1}^{N_j} \lambda^j_l(g_k)\, 2^{-j\frac{n-1}{2}}\, \Phi^j_l(\gamma') \\ &= g_k(\gamma'), \quad k=0,\ldots,r.\end{aligned} \tag{5.93}$$

Hence,

$$\mathrm{tr}^r_\Gamma \circ \mathrm{Ext}^{r,u}_\Gamma = \mathrm{id}, \quad \text{identity in } \prod_{k=0}^{r} B^{s-\frac{1}{p}-k}_{pq}(\Gamma). \tag{5.94}$$

Since the traces for spaces on $\mathbb{R}^n$ and on Ω are the same one obtains now (5.78), (5.80) for $\mathrm{ext}^{r,u}_\Gamma = \mathrm{re}_\Omega \circ \mathrm{Ext}^{r,u}_\Gamma$.

Step 2. It is well known that traces for $F^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\Omega)$ on Γ, if exist, are independent of q, hence they coincide with the traces of $B^s_{pp}(\mathbb{R}^n)$ and $B^s_{pp}(\Omega)$. This applies in particular to (5.62), (5.63). It is also covered by [T83], Theorem 3.3.3, p. 200, based on Fourier-analytical arguments. But the main point is not so much the smoothness of Γ but its porosity according to Definition 3.4 (i). With g as in (5.89) we re-write the first estimate in (5.91) as

$$\|g\,|B^s_{pq}(\mathbb{R}^n)\| \le c \sum_{k=0}^{r} \|\mu_k\,|b^s_{pq}(Q)\| \tag{5.95}$$

with $\mu_k = \{\mu^j_{kl}\}$ and the collection of cubes $Q = \{Q_{jl}\}$ as in (5.86). Since $q > \varepsilon$ in (5.79) and $L \ge n\big(\frac{1}{\varepsilon} - 1\big)$ in (5.73) one obtains by Theorem 1.7 and Proposition 5.3 that a^j_{kl} in (5.88) are also L_∞-normalised atoms in $F^s_{pq}(\mathbb{R}^n)$. The counterpart of (5.95) for the spaces $F^s_{pq}(\mathbb{R}^n)$ according to Theorem 1.7 and the L_∞-normalised atoms in (5.89) is given by

$$\|g\,|F^s_{pq}(\mathbb{R}^n)\| \le c \sum_{k=0}^{r} \|\mu_k\,|f^s_{pq}(Q)\| \tag{5.96}$$

with

$$\|\mu_k\,|f^s_{pq}(Q)\| = \Big\| \Big(\sum_{j,l} 2^{jsq}\, |\mu^j_{kl}\, \chi_{jl}(\cdot)|^q\Big)^{1/q} |L_p(\mathbb{R}^n)\Big\|, \tag{5.97}$$

where χ_{jl} is the characteristic function of Q_{jl} in (5.86). But this is the point where the specific nature of Q_{jl} comes in. It follows by [T06], Proposition 1.33, Example 1.35, p. 19, that

$$\|\mu_k\,|f^s_{pq}(Q)\| \sim \|\mu_k\,|f^s_{pp}(Q)\| \sim \|\mu_k\,|b^s_{pp}(Q)\|. \tag{5.98}$$

One may also consult [T06], Proposition 9.22, pp. 393–94, for this type of argument. This proves (5.79) by reduction to (5.78) with $q = p$. □

Remark 5.15. By [T83], Theorem 3.3.3, p. 200, and the references in [T83], Sections 2.7.2, 3.3.3, it is known since some 30 years that tr^r_Γ with (5.59)–(5.63) is a *retraction*. This means that there is a linear and bounded operator, called *co-retraction*, with (5.80) (in the above notation). In particular [T83], Theorem 3.3.3, coincides with [Tri78], Theorem 2.4.2, p. 105. Trace theorems for the classical Besov spaces $B^s_{pq}(\Omega)$ with $p > 1$ and Sobolev spaces $H^s_p(\Omega) = F^s_{p,2}(\Omega)$, $1 < p < \infty$, are known since the late 1950s and 1960s. This may be found in [BIN75], [Nik77], [T78]. The main aim of the above theorem is not so much to give a new proof of this well-known assertion but to construct explicitly extension operators in terms of wavelets. This will be used in what follows.

We recalled at the beginning of Section 3.2.3 what is meant by a complemented subspace of a given quasi-Banach space and a related projection. Let tr^r_Γ and $\mathrm{ext}^{r,u}_\Gamma$ be as in the above theorem. Then $P^{r,u}$,

$$P^{r,u} = \mathrm{ext}^{r,u}_\Gamma \circ \mathrm{tr}^r_\Gamma : A^s_{pq}(\Omega) \hookrightarrow A^s_{pq}(\Omega), \tag{5.99}$$

with $A^s_{pq}(\Omega) = B^s_{pq}(\Omega)$ as in (5.78) or $A^s_{pq}(\Omega) = F^s_{pq}(\Omega)$ as in (5.79). By (5.80) one has

$$(P^{r,u})^2 = \mathrm{ext}^{r,u}_\Gamma \circ \mathrm{tr}^r_\Gamma \circ \mathrm{ext}^{r,u}_\Gamma \circ \mathrm{tr}^r_\Gamma = P^{r,u}. \tag{5.100}$$

Hence $P^{r,u}$ is a projection of $A^s_{pq}(\Omega)$ onto its range, denoted by $P^{r,u}A^s_{pq}(\Omega)$.

Corollary 5.16. *Let $r \in \mathbb{N}_0$ and $u \in \mathbb{N}$ with $r < u$. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition 3.4* (iii) *or a bounded interval in $\mathbb{R}$ if $n = 1$. Let $A^s_{pq}(\Omega)$ be either $B^s_{pq}(\Omega)$ with* (5.78) *or $F^s_{pq}(\Omega)$ with* (5.79)*. Then $P^{r,u}$ according to* (5.99) *is a projection in $A^s_{pq}(\Omega)$ and $\mathrm{ext}^{r,u}_\Gamma$ is an isomorphic map onto the range of $P^{r,u}$,*

$$\left.\begin{aligned} \mathrm{ext}^{r,u}_\Gamma \prod_{k=0}^r B^{s-\frac{1}{p}-k}_{pq}(\Gamma) &= P^{r,u}B^s_{pq}(\Omega), \\ \mathrm{ext}^{r,u}_\Gamma \prod_{k=0}^r B^{s-\frac{1}{p}-k}_{pp}(\Gamma) &= P^{r,u}F^s_{pq}(\Omega) \end{aligned}\right\} \tag{5.101}$$

(*equivalent quasi-norms*).

Proof. By (5.100) it remains to prove that $\mathrm{ext}^{r,u}_\Gamma$ generates the indicated isomorphic map. Let $A^s_{pq}(\Omega) = B^s_{pq}(\Omega)$. We use temporarily the abbreviation

$$B^{s-\frac{1}{p}}_{pq}(\Gamma)^r = \prod_{k=0}^r B^{s-\frac{1}{p}-k}_{pq}(\Gamma), \tag{5.102}$$

naturally quasi-normed as in (5.58). Then one has by (5.78) and (5.80), (5.99),

$$\| \mathrm{ext}^{r,u}_\Gamma g \,|B^s_{pq}(\Omega)\| \leq c \,\|g \,|B^{s-\frac{1}{p}}_{pq}(\Gamma)^r\| \tag{5.103}$$

and

$$\mathrm{ext}_\Gamma^{r,u}\, g = P^{r,u}\big[\mathrm{ext}_\Gamma^{r,u}\, g\big] \in P^{r,u} B^s_{pq}(\Omega). \tag{5.104}$$

If $f \in P^{r,u} B^s_{pq}(\Omega)$, hence $f = P^{r,u} f$, then

$$f = \mathrm{ext}_\Gamma^{r,u}\, g \quad \text{with } g = \mathrm{tr}_\Gamma^r\, f \in B_{pq}^{s-\frac{1}{p}}(\Gamma)^r \tag{5.105}$$

and

$$\|g\,|B_{pq}^{s-\frac{1}{p}}(\Gamma)^r\| \le c\, \|f\,|B^s_{pq}(\Omega)\| = c\, \|\mathrm{ext}_\Gamma^{r,u}\, g\,|B^s_{pq}(\Omega)\|. \tag{5.106}$$

Then one obtains the isomorphic map (5.101) for the B-spaces. Similarly for the F-spaces. □

5.1.4 Decompositions

As indicated in Introduction 5.1.1 we wish to clip together the (orthonormal) u-wavelet bases for the spaces $\tilde{A}^s_{pq}(\Omega)$ according to Theorem 3.13 with the u-wavelet frames for related spaces on $\Gamma = \partial\Omega$ as described in Theorem 5.9 with Theorem 5.14 as the decisive vehicle. For this purpose we need the decompositions (5.12)–(5.15) which we are going to discuss now in greater detail. In the next Section 5.1.5 we combine all these ingredients to construct wavelet frames (and later on also wavelet bases) in the corresponding spaces $A^s_{pq}(\Omega)$. First we complement Definition 2.1.

Definition 5.17. Let Ω be an arbitrary domain in $\mathbb{R}^n$.

(i) Let p, q, s be as in (2.3) with $p < \infty$ for the F-spaces. Then $\mathring{A}^s_{pq}(\Omega)$ is the completion of $D(\Omega)$ in $A^s_{pq}(\Omega)$.

(ii) Let $u \in \mathbb{N}_0$. Then $C^u(\Omega)$ is the collection of all complex-valued functions f having classical derivatives up to order u inclusively in Ω such that any function $D^\alpha f$ with $|\alpha| \le u$ can be extended continuously to $\bar{\Omega}$ and

$$\|f\,|C^u(\Omega)\| = \sum_{|\alpha|\le u} \sup_{x\in\Omega} |D^\alpha f(x)| < \infty. \tag{5.107}$$

Furthermore $C(\Omega) = C^0(\Omega)$ and

$$C^\infty(\Omega) = \bigcap_{u=0}^{\infty} C^u(\Omega). \tag{5.108}$$

Remark 5.18. Whereas the notation $\mathring{A}^s_{pq}(\Omega)$ is in common use, the situation for spaces of type $C^u(\Omega)$ is different. The above version coincides with [HaT08], Definition A.1. Of course, $C^u(\Omega)$ is a Banach space. If Ω is a bounded C^∞ domain in $\mathbb{R}^n$ then $C^u(\Omega)$ can be recovered from $C^u(\mathbb{R}^n)$ by restriction in the same way as in Definition 2.1 for the spaces $A^s_{pq}(\Omega)$. We refer to [HaT08], Theorem 4.1, p. 89.

We are mainly interested in the relations between $\tilde{A}^s_{pq}(\Omega)$, $\mathring{A}^s_{pq}(\Omega)$, and subspaces of $A^s_{pq}(\Omega)$ with zero traces. All that we need is available in the literature.

Proposition 5.19. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, according to Definition 3.4* (iii) *or a bounded interval in $\mathbb{R}$ if $n = 1$. Let*

$$0 < p < \infty, \quad 0 < q < \infty, \quad \max\left(\frac{1}{p} - 1, n\left(\frac{1}{p} - 1\right)\right) < s < \frac{1}{p} \tag{5.109}$$

with $q \geq \min(p, 1)$ for the F-spaces. Then

$$\mathring{A}^s_{pq}(\Omega) = A^s_{pq}(\Omega) = \tilde{A}^s_{pq}(\Omega). \tag{5.110}$$

Remark 5.20. This coincides with a corresponding assertion in [T06], Section 1.11.6, p. 66, where we referred in turn to [Tri02], Proposition 3.1, p. 494. In [T06], p. 67, we quoted

$$\mathring{A}^s_{pq}(\Omega) = \tilde{A}^s_{pq}(\Omega), \quad 0 < p < \infty, \ 0 < q < \infty, \tag{5.111}$$

for bounded C^∞ domains Ω and

$$s > \sigma_p = n\left(\frac{1}{p} - 1\right)_+, \quad s - \frac{1}{p} \notin \mathbb{N}_0, \tag{5.112}$$

with a reference to [T01], p. 69–70, and the literature mentioned there. But this is a somewhat tricky assertion. We need only (5.110) with $1 \leq p < \infty$.

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ (a bounded interval in $\mathbb{R}$ if $n = 1$) and let $P^{r,u}$ be the same projection as in Corollary 5.16 and (5.99). Then $Q^{r,u} = \mathrm{id} - P^{r,u}$ is the projection of $A^s_{pq}(\Omega)$ onto the kernel of $P^{r,u}$,

$$Q^{r,u} A^s_{pq}(\Omega) = \left\{ f \in A^s_{pq}(\Omega) : \ \mathrm{tr}^r_\Gamma f = 0 \right\}. \tag{5.113}$$

If $b \in \mathbb{R}$ then $[b]^- \in \mathbb{Z}$ denotes the largest integer strictly less than b. If $p \geq 1$ in (5.78), (5.79) then $r = \left[s - \frac{1}{p}\right]^-$ is the largest admitted $r \in \mathbb{N}_0$. As before, the corresponding trace operator $\mathrm{tr}_\Gamma^{[s-\frac{1}{p}]^-}$ is given by (5.59),

$$\mathrm{tr}_\Gamma^{[s-\frac{1}{p}]^-} : f \mapsto \left\{ \mathrm{tr}_\Gamma \frac{\partial^j f}{\partial \nu^j} : \ 0 \leq j \leq \left[s - \tfrac{1}{p}\right]^- \right\}. \tag{5.114}$$

Let $\mathrm{ext}_\Gamma^{[s-\frac{1}{p}]^{-1},u}$ be the same as in (5.78), (5.79) with (5.101).

Theorem 5.21. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition 3.4* (iii) *or a bounded interval in $\mathbb{R}$ if $n = 1$. Let $\Gamma = \partial\Omega$. Let*

$$1 \leq p < \infty, \quad -1 < s - \frac{1}{p} \notin \mathbb{N}_0, \quad \begin{cases} 0 < q < \infty, & B\text{-spaces}, \\ 1 \leq q < \infty, & F\text{-spaces}, \end{cases} \tag{5.115}$$

and $u \in \mathbb{N}$ with $s < u$. Then

$$\left.\begin{aligned} \widetilde{B}^s_{pq}(\Omega) &= \overset{\circ}{B}{}^s_{pq}(\Omega) = \big\{ f \in B^s_{pq}(\Omega) : \operatorname{tr}_\Gamma^{[s-\frac{1}{p}]^-} f = 0 \big\}, \\ \widetilde{F}^s_{pq}(\Omega) &= \overset{\circ}{F}{}^s_{pq}(\Omega) = \big\{ f \in F^s_{pq}(\Omega) : \operatorname{tr}_\Gamma^{[s-\frac{1}{p}]^-} f = 0 \big\} \end{aligned}\right\} \tag{5.116}$$

interpreted as (5.110) if $-1 + \frac{1}{p} < s < \frac{1}{p}$. Furthermore,

$$\left.\begin{aligned} B^s_{pq}(\Omega) &= \widetilde{B}^s_{pq}(\Omega) \times \operatorname{ext}_\Gamma^{[s-\frac{1}{p}]^-,u} \prod_{k=0}^{[s-\frac{1}{p}]^-} B^{s-\frac{1}{p}-k}_{pq}(\Gamma), \\ F^s_{pq}(\Omega) &= \widetilde{F}^s_{pq}(\Omega) \times \operatorname{ext}_\Gamma^{[s-\frac{1}{p}]^-,u} \prod_{k=0}^{[s-\frac{1}{p}]^-} B^{s-\frac{1}{p}-k}_{pp}(\Gamma). \end{aligned}\right\} \tag{5.117}$$

Proof. By Proposition 5.19 we may assume $s > \frac{1}{p}$. Let $r = \big[s - \frac{1}{p}\big]^- = \big[s - \frac{1}{p}\big]$. Then it follows from (5.101) and (5.113) that $B^s_{pq}(\Omega)$ can be decomposed as

$$B^s_{pq}(\Omega) = \big\{ f \in B^s_{pq}(\Omega) : \operatorname{tr}^r_\Gamma f = 0 \big\} \times \operatorname{ext}^{r,u}_\Gamma \prod_{k=0}^{r} B^{s-\frac{1}{p}-k}_{pq}(\Gamma) \tag{5.118}$$

with

$$\| f \,|B^s_{pq}(\Omega)\| \sim \|P^{r,u} f \,|B^s_{pq}(\Omega)\| + \|Q^{r,u} f \,|B^s_{pq}(\Omega)\|. \tag{5.119}$$

By [T83], Theorem and Corollary 3.4.3, p. 210, and the related proofs one has

$$\big\{ f \in B^s_{pq}(\Omega) : \operatorname{tr}^r_\Gamma f = 0 \big\} = \overset{\circ}{B}{}^s_{pq}(\Omega) = \widetilde{B}^s_{pq}(\Omega) \tag{5.120}$$

and an F-counterpart. This proves (5.116) and also the B-part of (5.117). In case of the F-spaces one has to use the F-part of (5.101). □

Remark 5.22. Quite obviously, (5.117) is the precise version of (5.12)–(5.15). Somewhat in contrast to Theorem 5.14 and Corollary 5.16 we restricted p in Theorem 5.21 to $p \geq 1$. By the above arguments and [T83], Theorem and Corollary 3.4.3, p. 210, one can extend Theorem 5.21 to $A^s_{pq}(\Omega)$ with

$$0 < p < 1, \quad n\Big(\frac{1}{p} - 1\Big) < s - r < \frac{1}{p}, \quad \begin{cases} 0 < q < \infty, & B\text{-spaces}, \\ p \leq q < \infty, & F\text{-spaces}, \end{cases} \tag{5.121}$$

where $r \in \mathbb{N}_0$. In contrast to the above theorem where for $p \geq 1$ only the values $s - \frac{1}{p} \in \mathbb{N}_0$ are not covered, one must exclude for $p < 1$ large s-intervals. This is not very satisfactory, but unavoidable. In Section 6.4 we discuss which curious effects can happen for traces of spaces $A^s_{pq}(\mathbb{R}^n)$ if $p < 1$. On the other hand, one can replace $1 \leq q < \infty$ in (5.115) for the F-spaces by $0 < q < \infty$. We refer to Corollary 6.24

and Remark 6.25. We add some comments about the spaces $A^s_{pq}(\Omega)$ with $s - \frac{1}{p} \in \mathbb{N}_0$ making clear that assertions of type (5.110), (5.111) and (5.116), (5.117) cannot be expected. Let again Ω be a bounded C^∞ in $\mathbb{R}^n$ and let $0 < p < \infty$, $0 < q < \infty$. Then it follows from [Tri07d], also subject of Section 6.4 below, that

$$\left.\begin{aligned} \mathring{B}^{1/p}_{pq}(\Omega) &= B^{1/p}_{pq}(\Omega) \quad \text{if, and only if, } \min(1,p) < q < \infty, \\ \mathring{F}^{1/p}_{pq}(\Omega) &= F^{1/p}_{pq}(\Omega) \quad \text{if, and only if, } p > 1,\ 0 < q < \infty. \end{aligned}\right\} \tag{5.122}$$

On the other hand, according to [T01], Sections 5.7, 5.9, pp. 51, 53, and the references given there one has for

$$0 < p < \infty, \quad 0 < q < \infty, \quad s > \sigma_p = n\Big(\frac{1}{p} - 1\Big)_+ \tag{5.123}$$

the *Hardy inequalities*

$$\int_\Omega \frac{|f(x)|^p}{d^{sp}(x)}\, \mathrm{d}x \le c\, \|f\,|F^s_{pq}(\mathbb{R}^n)\|^p, \quad f \in \widetilde{F}^s_{pq}(\Omega), \tag{5.124}$$

where again $d(x) = \mathrm{dist}(x,\Gamma)$ is the distance of $x \in \Omega$ to Γ, and

$$\int_0^\infty t^{-sq}\Big(\int_{\Omega^t} |f(x)|^p\, \mathrm{d}x\Big)^{q/p} \frac{\mathrm{d}t}{t} \le c\, \|f\,|B^s_{pq}(\mathbb{R}^n)\|^q, \quad f \in \widetilde{B}^s_{pq}(\Omega), \tag{5.125}$$

with $\Omega^t = \{x \in \Omega : d(x) < t\}$. It follows that the characteristic function of Ω does not belong to any of the spaces $\tilde{A}^{1/p}_{pq}(\Omega)$ with (5.123), hence

$$\tilde{A}^{1/p}_{pq}(\Omega) \neq A^{1/p}_{pq}(\Omega), \quad 0 < p < \infty,\ 0 < q < \infty,\ \tfrac{1}{p} > \sigma_p. \tag{5.126}$$

This shows that for $s = 1/p$ nothing like (5.110), (5.116) and (5.117) with empty second factors on the right-hand side, can be expected in cases covered by (5.122), (5.126). Let

$$0 < p < \infty, \quad 0 < q < \infty, \quad s = r + \frac{1}{p} > \sigma_p \quad \text{and} \quad r \in \mathbb{N}. \tag{5.127}$$

Then one has

$$\left.\begin{aligned} \mathring{B}^s_{pq}(\Omega) &\neq \widetilde{B}^s_{pq}(\Omega) \quad \text{if } \min(1,p) < q < \infty, \\ \mathring{F}^s_{pq}(\Omega) &\neq \widetilde{F}^s_{pq}(\Omega) \quad \text{if } p > 1,\ 0 < q < \infty. \end{aligned}\right\} \tag{5.128}$$

This follows from

$$g \in \mathring{A}^{r+\frac{1}{p}}_{pq}(\Omega) \quad \text{and} \quad g \notin \tilde{A}^{r+\frac{1}{p}}_{pq}(\Omega) \tag{5.129}$$

with $g \in C^\infty(\Omega)$ according to (5.108) and $g(x) \sim d^r(x)$ near Γ. One obtains the second assertion in (5.129) from (5.124), (5.125) with $s = r + \frac{1}{p}$. The first assertion is covered by the arguments in [Tri07d]. This makes clear that assertions of type

(5.116) cannot be expected for the exceptional cases $s = r + \frac{1}{p}$ with $r \in \mathbb{N}_0$. As a consequence we prove in Section 6.2.2 that the spaces $\mathring{A}^s_{pq}(\Omega)$ in (5.127), (5.128) do not have an (interior) wavelet frame or basis in contrast to the corresponding spaces $\tilde{A}^s_{pq}(\Omega)$ according to Theorem 3.13. Assertions of type (5.126), (5.122) and (5.128) for related classical Besov spaces B^s_{pq} with $1 < p, q < \infty$ and Sobolev spaces $H^s_p = F^s_{p,2}$ with $1 < p < \infty$ are known since the 1960s and 1970s. This may be found in [T78], Section 4.3.2, pp. 317–20, and the references given there.

5.1.5 Wavelet frames in domains

As far as wavelet frames are concerned we complete now the programme outlined in Introduction 5.1.1 and recalled at the beginning of Section 5.1.4. We need the counterparts of the u-wavelet system $\{\Phi^j_r\}$ in Definition 2.4 and of the sequence spaces in Definition 2.6.

Definition 5.23. Let Ω be a bounded domain in $\mathbb{R}^n$ and let

$$\mathbb{Z}^\Omega = \{x^j_l \in \Omega : \; j \in \mathbb{N}_0; \; l = 1, \ldots, N_j\}, \tag{5.130}$$

typically with $N_j \sim 2^{jn}$, such that for some $c_1 > 0$,

$$|x^j_l - x^j_{l'}| \geq c_1 \, 2^{-j}, \quad j \in \mathbb{N}_0, \; l \neq l'. \tag{5.131}$$

Let χ_{jl} be the characteristic function of the ball $B(x^j_l, c_2 \, 2^{-j}) \subset \mathbb{R}^n$ (centred at x^j_l and of radius $c_2 \, 2^{-j}$) for some $c_2 > 0$. Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. Then $b^s_{pq}(\mathbb{Z}^\Omega)$ is the collection of all sequences

$$\lambda = \{\lambda^j_l \in \mathbb{C} : \; j \in \mathbb{N}_0; \; l = 1, \ldots, N_j\} \tag{5.132}$$

such that

$$\|\lambda \,|b^s_{pq}(\mathbb{Z}^\Omega)\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \Big(\sum_{l=1}^{N_j} |\lambda^j_l|^p \Big)^{q/p} \Big)^{1/q} < \infty, \tag{5.133}$$

and $f^s_{pq}(\mathbb{Z}^\Omega)$ is the collection of all sequences (5.132) such that

$$\|\lambda \,| f^s_{pq}(\mathbb{Z}^\Omega)\| = \Big\| \Big(\sum_{j=0}^{\infty} \sum_{l=1}^{N_j} 2^{jsq} \, |\lambda^j_l \, \chi_{jl}(\cdot)|^q \Big)^{1/q} |L_p(\Omega) \Big\| < \infty \tag{5.134}$$

(obviously modified if $p = \infty$ and/or $q = \infty$).

Remark 5.24. Roughly speaking, $\mathbb{Z}^\Omega$ complements $\mathbb{Z}_\Omega$ according to (2.24)–(2.26) by points near the boundary. This is the almost obvious modification of Definition 2.6.

In what follows we are interested in bounded C^∞ domains and bounded Lipschitz domains. If $p < \infty$ then one can replace χ_{jl} in (5.134) by the characteristic function of $B(x^j_l, c'_2 2^{-j}) \cap \Omega$ with $c'_2 > 0$ (equivalent quasi-norms). This is well known and covered by [T06], Section 1.5.3, pp. 18–19, and the references given there.

Recall that in this Chapter 5 we are mainly interested in spaces $A^s_{pq}(\Omega)$ according to (5.3) where Ω is a bounded C^∞ domain. Then related atoms as described in Definition 1.5, Theorem 1.7 need not to have moment conditions. But it remains to be a desirable property, also in connection with numerical applications in terms of polynomial approximations. The interior wavelets in (2.33) have this property, for basic wavelets (2.32) it is not needed, and for the boundary wavelets (2.34) not necessarily valid. This can be preserved in what follows. But this has to be complemented now by wavelets of type (5.76). The product structure of these building blocks combined with a mild modification results in wavelets with some qualitative cancellations of the same type as in (5.32). Then we speak about *oscillation* instead of cancellation. This may justify to introduce wavelets in $\bar{\Omega}$ in modification of Definitions 2.4, 5.5. The spaces $C^u(\Omega)$ have the same meaning as in Definition 5.17.

Definition 5.25. Let Ω be a bounded domain in $\mathbb{R}^n$ with $\Gamma = \partial\Omega$ and let $\mathbb{Z}^\Omega$ be as in (5.130), (5.131). Let $u \in \mathbb{N}$.

(i) Then

$$\Phi = \left\{ \Phi^j_l : j \in \mathbb{N}_0;\ l = 1, \ldots, N_j \right\} \subset C^u(\Omega) \tag{5.135}$$

is called a u-wavelet system in $\bar{\Omega}$ if for some $c_3 > 0$ and $c_4 > 0$,

$$\operatorname{supp} \Phi^j_l \subset B(x^j_l, c_3 2^{-j}) \cap \bar{\Omega}, \quad j \in \mathbb{N}_0;\ l = 1, \ldots, N_j, \tag{5.136}$$

and

$$\left| D^\alpha \Phi^j_l(x) \right| \le c_4 \, 2^{j\frac{n}{2} + j|\alpha|}, \quad j \in \mathbb{N}_0;\ l = 1, \ldots, N_j,\ x \in \Omega, \tag{5.137}$$

for $\alpha \in \mathbb{N}^n_0$ with $0 \le |\alpha| \le u$.

(ii) The above u-wavelet system is called oscillating if there are positive numbers c_5, c_6, c_7 with $c_6 < c_7$ such that

$$\operatorname{dist}(B(x^0_l, c_3), \Gamma) \ge c_6, \quad l = 1, \ldots, N_0, \tag{5.138}$$

and

$$\left| \int_\Omega \psi(x)\, \Phi^j_l(x)\, \mathrm{d}x \right| \le c_5 \, 2^{-j\frac{n}{2} - ju} \, \|\psi \,|C^u(\Omega)\|, \quad \psi \in C^u(\Omega), \tag{5.139}$$

for all Φ^j_l with $j \in \mathbb{N}$ and

$$\operatorname{dist}(B(x^j_l, c_3 2^{-j}), \Gamma) \notin (c_6 2^{-j}, c_7 2^{-j}). \tag{5.140}$$

Remark 5.26. In other words, the oscillation condition (5.139) is not required for the terms with $j = 0$, for which we assume (5.138), and for wavelets Φ^j_l with $j \in \mathbb{N}$ and, roughly speaking,

$$\operatorname{dist}(x^j_l, \Gamma) \sim \operatorname{dist}(\operatorname{supp} \Phi^j_l, \Gamma) \sim 2^{-j}. \tag{5.141}$$

These are the basic wavelets Φ^0_l in (2.32) and the boundary wavelets Φ^j_l in (2.34) with (2.26), (2.35). On the other hand, one requires (5.139) in particular for all wavelets Φ^j_l with supports having non-empty intersection with Γ.

As before we use $A^s_{pq}(\Omega)$ with $A \in \{B, F\}$ and correspondingly $a^s_{pq}(\mathbb{Z}^\Omega)$ with $a \in \{b, f\}$ according to Definitions 2.1 and 5.23. Equivalences $\sim$ must be understood as in (3.4), (3.5).

Theorem 5.27. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition 3.4* (iii) *or a bounded interval in $\mathbb{R}$ if $n = 1$. Let $r \in \mathbb{N}_0$ and $u \in \mathbb{N}$ with $r < u$. Then there is a common oscillating u-wavelet system Φ_r according to Definition 5.25* (ii) *for all p, q, s with*

$$1 \leq p < \infty, \quad \frac{1}{p} - 1 < s - r < \frac{1}{p}, \quad \begin{cases} 0 < q < \infty, & B\text{-spaces}, \\ 1 \leq q < \infty, & F\text{-spaces}, \end{cases} \tag{5.142}$$

and $A \in \{B, F\}$ such that an element $f \in D'(\Omega)$ belongs to $A^s_{pq}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda^j_l \, 2^{-jn/2} \, \Phi^j_l, \quad \lambda \in a^s_{pq}(\mathbb{Z}^\Omega), \tag{5.143}$$

unconditional convergence being in $A^s_{pq}(\Omega)$. Furthermore,

$$\| f \,|A^s_{pq}(\Omega)\| \sim \inf \|\lambda \,|a^s_{pq}(\mathbb{Z}^\Omega)\|, \tag{5.144}$$

where the infimum is taken over all admissible representations (5.143) (*equivalent quasi-norms*). *Any $f \in A^s_{pq}(\Omega)$ can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda^j_l(f) \, 2^{-jn/2} \, \Phi^j_l \tag{5.145}$$

where $\lambda^j_l(\cdot) \in A^s_{pq}(\Omega)'$ are linear and continuous functionals on $A^s_{pq}(\Omega)$ and

$$\| f \,|A^s_{pq}(\Omega)\| \sim \|\lambda(f) \,|a^s_{pq}(\mathbb{Z}^\Omega)\| \tag{5.146}$$

(*u-wavelet frame*).

Proof. Step 1. Let $r = 0$ and $u \in \mathbb{N}$. Then $p > 1$ if $s \le 0$. The orthonormal u-wavelet basis as used in Theorem 3.23 is oscillating. Then one obtains the desired assertion from this theorem and Proposition 5.19.

Step 2. Let $r \in \mathbb{N}$ and $u \in \mathbb{N}$ with $u > r$. First we construct a suitable oscillating u-wavelet system in $\overline{\Omega}$ consisting of the interior orthonormal u-wavelet basis in $L_2(\Omega)$ as used in Theorem 3.13, now denoted by

$$\Phi^{\Omega} = \{\Phi_l^{j,\Omega} : j \in \mathbb{N}_0;\ l = 1, \dots, N_j^{\Omega}\}, \quad N_j^{\Omega} \in \mathbb{N}, \tag{5.147}$$

(where $N_j^{\Omega} \sim 2^{jn}$) and a boundary u-wavelet system

$$\Phi^{\Gamma,r} = \{\Phi_l^{j,k} : j \in \mathbb{N};\ k = 0, \dots, r-1;\ l = 1, \dots, N_j^{\Gamma}\} \tag{5.148}$$

with $\Gamma = \partial\Omega$ (where $N_j^{\Gamma} \sim 2^{j(n-1)}$) specified below. By Definition 2.4 and Remark 5.26 the system Φ^{Ω} is oscillating and fits in our scheme. The construction of $\Phi^{\Gamma,r}$ relies on the decomposition (5.117) with

$$\Big[s - \frac{1}{p}\Big]^{-} = r - 1, \quad r \in \mathbb{N}, \tag{5.149}$$

and (5.77), (5.75) now with $r-1$ in place of r. Furthermore we adapt (5.74)–(5.76) notationally to Definition 5.25 (ii) by shifting j to $j+1$, hence $j \in \mathbb{N}$ in place of $j \in \mathbb{N}_0$. We give an explicit formulation. Indicating now $\Gamma = \partial\Omega$ we let again

$$\Phi^{\Gamma} = \{\Phi_l^{j,\Gamma}\} \quad \text{and} \quad \Psi^{\Gamma} = \{\Psi_l^{j,\Gamma}\} \tag{5.150}$$

be two real u-wavelet systems according to Theorem 5.9 on the compact $(n-1)$-dimensional manifold Γ, where now $j \in \mathbb{N}$. Then

$$\lambda_{l,\Gamma}^{j}(h) = 2^{j\frac{n-1}{2}} \big(h, \Psi_l^{j-1,\Gamma}\big)_{\Gamma} = 2^{j\frac{n-1}{2}} \int_{\Gamma} h(\gamma')\, \Psi_l^{j-1,\Gamma}(\gamma')\, \mu(\mathrm{d}\gamma') \tag{5.151}$$

is the adapted counterpart of (5.74), $j \in \mathbb{N}$. Similarly we replace (5.75), (5.76) by

$$\begin{aligned} g &= \mathrm{Ext}_{\Gamma}^{r-1,u}\{g_0, \dots, g_{r-1}\} \\ &= \sum_{k=0}^{r-1} \sum_{j=1}^{\infty} \sum_{l=1}^{N_j^{\Gamma}} \frac{1}{k!}\, \lambda_{l,\Gamma}^{j}(g_k)\, 2^{-jk}\, 2^{-jn/2}\, \Phi_l^{j,k}(\gamma) \end{aligned} \tag{5.152}$$

with

$$\Phi_l^{j,k}(\gamma) = 2^{jk}\, \gamma_n^k\, \chi(2^j \gamma_n)\, 2^{j/2}\, \Phi_l^{j-1,\Gamma}(\gamma'). \tag{5.153}$$

We choose now χ in (5.72) such that

$$\int_0^{\infty} \chi(t)\, t^l\, \mathrm{d}t = 0, \quad l = 0, \dots, u-1. \tag{5.154}$$

Recall that $\gamma = (\gamma', \gamma_n)$ are C^∞ curvilinear coordinates in a tubular neighbourhood of Γ. Then it follows that one has (5.136), (5.137) and (5.139) with $\Phi_l^{j,k}$ in place of Φ_l^j. With $\Phi^{\Gamma,r}$ as in (5.148) based on (5.153) one obtains that

$$\Phi_r = \{\Phi_l^j : j \in \mathbb{N}_0;\ l = 1, \ldots, N_j\} = \Phi^\Omega \cup \Phi^{\Gamma,r}, \quad N_j \sim 2^{jn}, \tag{5.155}$$

is an oscillating u-wavelet system according to Definition 5.25 (ii). According to Theorem 5.14 and its proof, (5.152) is an extension operator

$$\mathrm{Ext}_\Gamma^{r-1,u} : \quad \prod_{k=0}^{r-1} B_{pq}^{s-\frac{1}{p}-k}(\Gamma) \hookrightarrow B_{pq}^s(\mathbb{R}^n) \tag{5.156}$$

and

$$\begin{aligned} \|g \,|B_{pq}^s(\mathbb{R}^n)\| &\le c \sum_{k=0}^{r-1} \big\|g_k \,|B_{pq}^{s-\frac{1}{p}-k}(\Gamma)\big\| \\ &\sim c' \sum_{k=0}^{r-1} \Big(\sum_{j=0}^\infty 2^{j(s-\frac{n}{p}-k)q} \Big(\sum_{l=1}^{N_j^\Gamma} |\lambda_{l,\Gamma}^j(g_k)|^p \Big)^{q/p} \Big)^{1/q}. \end{aligned} \tag{5.157}$$

Step 3. We prove that the oscillating u-wavelet system Φ_r in (5.155) has the desired properties. We deal with the B-spaces. The necessary modifications for the F-spaces are the same as in Step 2 of the proof of Theorem 5.14 where now the additional assumption (5.73) with $\varepsilon = 1$ and $L = 0$ is empty. Let $f \in D'(\Omega)$ be given by (5.143) with $a = b$. Then $2^{-jn/2}\,\Phi_l^j$ with either $\Phi_l^j = \Phi_l^{j,\Omega}$ or $\Phi_l^j = \Phi_l^{j,k}$ are L_∞-normalised atoms in $\mathbb{R}^n$ (no moment conditions are needed). We have by Theorem 1.7 that $f \in B_{pq}^s(\mathbb{R}^n)$ and

$$\|\,\mathrm{re}_\Omega f \,|B_{pq}^s(\Omega)\| \le \|f \,|B_{pq}^s(\mathbb{R}^n)\| \le c\, \|\lambda \,|b_{pq}^s(\mathbb{Z}^\Omega)\|. \tag{5.158}$$

It remains to prove that any $f \in B_{pq}^s(\Omega)$ can be represented by (5.145) with Φ_l^j according to (5.155) with (5.147) and (5.148), (5.153). We decompose $f \in B_{pq}^s(\Omega)$ by

$$f = P^{r,u} f + Q^{r,u} f \tag{5.159}$$

with (5.119). By (5.113) and (5.120) one can expand $Q^{r,u} f$ according to Theorem 3.13 by the system Φ^Ω in (5.147),

$$Q^{r,u} f = \sum_{j=0}^\infty \sum_{l=1}^{N_j} \lambda_l^{j,\Omega}(Q^{r,u} f)\, 2^{-jn/2}\, \Phi_l^{j,\Omega} \tag{5.160}$$

with

$$\lambda_l^{j,\Omega}(Q^{r,u} f) = 2^{jn/2} \int_\Omega (Q^{r,u} f)(x)\, \Phi_l^{j,\Omega}(x)\, \mathrm{d}x \tag{5.161}$$

and

$$\|\lambda^{\Omega}(Q^{r,u} f) \,|b^s_{pq}(\mathbb{Z}_\Omega)\| \sim \|Q^{r,u} f \,|B^s_{pq}(\Omega)\| \le c\, \|f \,|B^s_{pq}(\Omega)\|. \tag{5.162}$$

In modification of (5.99) one obtains by (5.152) and (5.151) that

$$P^{r,u} f = \sum_{k=0}^{r-1} \sum_{j=1}^{\infty} \sum_{l=1}^{N_j^\Gamma} \lambda_l^{j,\Gamma} \Big(\operatorname{tr}_\Gamma \frac{\partial^k f}{\partial \nu^k} \Big) 2^{-jn/2}\, \Phi_l^{j,k}(\gamma) \tag{5.163}$$

with

$$\lambda_l^{j,\Gamma} \Big(\operatorname{tr}_\Gamma \frac{\partial^k f}{\partial \nu^k} \Big) = \frac{1}{k!}\, \lambda^j_{l,\Gamma} \Big(\operatorname{tr}_\Gamma \frac{\partial^k f}{\partial \nu^k} \Big) 2^{-jk}. \tag{5.164}$$

Recall that Γ is considered as an $(n-1)$-dimensional compact C^∞ manifold. We may identify the boundary part of $\mathbb{Z}^\Omega$ with $\mathbb{Z}_\Gamma$ in (5.34) with $n-1$ in place of n. Then one obtains by (5.164),

$$\begin{aligned} &\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \Big(\sum_{l=1}^{N_j^\Gamma} \Big| \lambda_l^{j,\Gamma} \Big(\operatorname{tr}_\Gamma \frac{\partial^k f}{\partial \nu^k} \Big) \Big|^p \Big)^{q/p} \\ &\quad \sim \sum_{j=0}^{\infty} 2^{j(s-k-\frac{1}{p}-\frac{n-1}{p})q} \Big(\sum_{l=1}^{N_j^\Gamma} \Big| \lambda^j_{l,\Gamma} \Big(\operatorname{tr}_\Gamma \frac{\partial^k f}{\partial \nu^k} \Big) \Big|^p \Big)^{q/p} \\ &\quad \sim \Big\| \operatorname{tr}_\Gamma \frac{\partial^k f}{\partial \nu^k} \,|B_{pq}^{s-k-\frac{1}{p}}(\Gamma) \Big\|^q \\ &\quad \le c\, \|f \,|B^s_{pq}(\Omega)\|^q, \end{aligned} \tag{5.165}$$

where we used Theorem 5.9 and (5.59), (5.60). Now one obtains by (5.159), (5.160), (5.163) and (5.162), (5.165) the representation (5.145) with

$$\|\lambda(f) \,|b^s_{pq}(\mathbb{Z}^\Omega)\| \le c\, \|f \,|B^s_{pq}(\Omega)\|. \tag{5.166}$$

This proves also (5.146). By construction the coefficients $\lambda^j_l(f)$ are linear continuous functionals on $B^s_{pq}(\Omega)$. □

Remark 5.28. We explained in Remark 5.11 what is meant by a *frame*. This justifies to call also the u-wavelet system $\{\Phi^j_l\}$ in the above theorem a frame characterised by the *stability* (5.144) and the *optimality* (5.146). The question arises whether this frame is even a basis. This can be reduced to a corresponding question for the frames on Γ according to Theorem 5.9. As mentioned in Remark 5.12 we deal with these problems in Sections 5.2, 5.3. The dual system $\{\Psi^j_l\}$ of the wavelet frame $\{\Phi^j_l\}$ for compact C^∞ manifolds in Theorem 5.9 is again a wavelet system of the same type. Nothing like this can be expected for the system $\{\Phi^j_l\}$ in the above theorem. In case of the boundary functionals $\lambda^j_l(f)$ originating from (5.164) it follows from (5.165) with $\mathbb{R}^n$ in place of Ω and (5.151) that

$$\lambda^j_l(f) = (f, \Psi^j_l) \quad \text{with } \Psi^j_l \in B^{-s}_{p'q'}(\mathbb{R}^n), \ \operatorname{supp} \Psi^j_l \subset \Gamma, \tag{5.167}$$

in the framework of the dual pairing $(S(\mathbb{R}^n), S'(\mathbb{R}^n))$. In particular, Ψ^j_l is a singular distribution.

Remark 5.29. In Theorems 3.13 and 3.23 we described common wavelet bases for large regions $\{s, p, q\}$ and $A \in \{B, F\}$. By Theorem 5.9 we have a similar assertion for spaces $A^s_{pq}(\Gamma)$ on compact C^∞ manifolds now in terms of wavelet frames. The wavelet frames in the above Theorem 5.27 depend on $r \in \mathbb{N}_0$ and they are restricted to the (s, p)-strips in (5.142). The problems for spaces $A^s_{pq}(\Omega)$ with $s - \frac{1}{p} \in \mathbb{N}_0$ or $p < 1$ are the same as in Remark 5.22.

The above theorem can be extended to some subspaces

$$A^s_{pq}(\Omega)^{\bar{r}} = \left\{ f \in A^s_{pq}(\Omega) : \ \operatorname{tr}_\Gamma \frac{\partial^j f}{\partial \nu^j} = 0; \ j = r_1, \dots, r_k \right\} \tag{5.168}$$

of $A^s_{pq}(\Omega)$ with

$$\bar{r} = \{r_1, \dots, r_k\} \subset \{0, \dots, r-1\}, \quad r \in \mathbb{N},$$

being integers between 0 and $r-1$, where p, q, s and $A \in \{B, F\}$ as in Theorem 5.27. Then $A^s_{pq}(\Omega)^{\bar{r}}$ is a complemented subspace of $A^s_{pq}(\Omega)$. Let $\bar{t} = \{t_1, \dots, t_m\}$ be the complementary integers between 0 and $r-1$, hence

$$\{0, \dots, r-1\} = \{r_1, \dots, r_k\} \cup \{t_1, \dots, t_m\}, \quad k \in \mathbb{N}, \ m \in \mathbb{N}, \ k + m = r, \tag{5.169}$$

excluding $\tilde{A}^s_{pq}(\Omega)$ and $A^s_{pq}(\Omega)$ in

$$\tilde{A}^s_{pq}(\Omega) \subset A^s_{pq}(\Omega)^{\bar{r}} \subset A^s_{pq}(\Omega). \tag{5.170}$$

Corollary 5.30. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition 3.4* (iii) *or a bounded interval in $\mathbb{R}$ if $n = 1$. Let $r \in \mathbb{N}$ and $u \in \mathbb{N}$ with $r < u$. Let p, q, s be as in* (5.142) *and $A \in \{B, F\}$. Let $\bar{r} = \{r_1, \dots, r_k\}$ be as in* (5.169)*. Then there is a common oscillating u-wavelet system according to Definition 5.25* (ii) *depending only on $\bar{r}$ and u such that an element $f \in A^s_{pq}(\Omega)$ belongs to $A^s_{pq}(\Omega)^{\bar{r}}$ if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda^j_l \, 2^{-jn/2} \, \Phi^j_l, \quad \lambda \in a^s_{pq}(\mathbb{Z}^\Omega), \tag{5.171}$$

unconditional convergence being in $A^s_{pq}(\Omega)$. Furthermore,

$$\| f \, | A^s_{pq}(\Omega) \| \sim \inf \| \lambda \, | a^s_{pq}(\mathbb{Z}^\Omega) \| \tag{5.172}$$

where the infimum is taken over all admissible representations (5.171) (*equivalent quasi-norms*)*. Any* $f \in A^s_{pq}(\Omega)^{\bar r}$ *can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda^j_l(f)\, 2^{-jn/2}\, \Phi^j_l, \tag{5.173}$$

where $\lambda^j_l(\cdot) \in A^s_{pq}(\Omega)'$ *are linear and continuous functionals on* $A^s_{pq}(\Omega)$ *and*

$$\|f\,|A^s_{pq}(\Omega)\| \sim \|\lambda(f)\,|a^s_{pq}(\mathbb{Z}^\Omega)\| \tag{5.174}$$

(*u-wavelet frame*).

Proof. We indicate the necessary modifications of the proof of Theorem 5.27 restricting us to the B-spaces. Let $\{t_1, \ldots, t_m\}$ be as in (5.169). Then the extension operator in (5.152) must be replaced now by

$$\mathrm{Ext}^{\bar r,u}_\Gamma\{g_{t_1}, \ldots, g_{t_m}\} = \sum_{v=1}^{m} \sum_{j=1}^{\infty} \sum_{l=1}^{N^\Gamma_j} \frac{1}{t_v!}\, \lambda^j_{l,\Gamma}(g_{t_v})\, 2^{-jt_v}\, 2^{-jn/2}\, \Phi^{j,t_v}_l(\gamma) \tag{5.175}$$

with (5.153). With an appropriate modification of $\Phi^{\Gamma,r}$ in (5.148) one can now follow the arguments from the proof of Theorem 5.27. □

Remark 5.31. Spaces of type (5.168) play a role in boundary value problems for elliptic operators. It remains to be seen whether one can take advantage of the existence of oscillating u-wavelet frames in this context.

5.2 Wavelet bases: criterion and lower dimensions

5.2.1 Wavelet bases on manifolds

One may ask under which circumstances the u-wavelet frame Φ_r in Theorem 5.27 is not only a frame but a unconditional basis as describe in (1.100), (1.101) in the respective spaces. Recall that Φ_r is given by (5.155) consisting of Φ^Ω in (5.147) and $\Phi^{\Gamma,r}$ in (5.148) (depending on r and u). Whereas Φ^Ω is the interior orthonormal u-wavelet basis in $L_2(\Omega)$ as used in Theorem 3.13, the system $\Phi^{\Gamma,r}$ is given by (5.153) based on the u-wavelet system Φ^Γ in (5.150) on $\Gamma = \partial\Omega$. It turns out that the u-wavelet frame in Theorem 5.27, given by (5.155), is a basis if, and only if, the underlying u-wavelet frame Φ^Γ according to (5.150) with a reference to Theorem 5.9 is a basis. This will be justified in Sections 5.2.1, 5.2.2 and applied to concrete situations afterwards. But first we return to abstract compact n-dimensional C^∞ manifolds and to the u-wavelet frame

$$\Phi^\Gamma = \left\{\Phi^{j,\Gamma}_l : j \in \mathbb{N}_0 : l = 1, \ldots, N^\Gamma_j\right\} \subset C^u(\Gamma) \tag{5.176}$$

according to Theorem 5.9 (notationally adapted to our later needs). If, in addition, Φ^Γ is a basis in $L_2(\Gamma)$, then (5.41), (5.42) can be written as

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j^\Gamma} \big(f, \Psi_l^{j,\Gamma} \big)_\Gamma \, \Phi_l^{j,\Gamma}, \quad f \in L_2(\Gamma), \tag{5.177}$$

with

$$\| f \, | L_2(\Gamma) \| \sim \Big(\sum_{j,l} \big| \big(f, \Psi_l^{j,\Gamma} \big)_\Gamma \big|^2 \Big)^{1/2}. \tag{5.178}$$

Hence Φ^Γ is a *Riesz basis* in $L_2(\Gamma)$. Recall that we may assume that $\Phi_l^{j,\Gamma}$ and $\Psi_l^{j,\Gamma}$ are L_2-normalised. A discussion about Riesz bases in (separable complex) Hilbert spaces may be found in [Gro01], p. 90. In particular, Φ^Γ is the isomorphic image of the standard basis in ℓ_2. Inserting $f = \Phi_l^{j,\Gamma}$ in (5.177) one obtains from the uniqueness the well-known property

$$\big(\Phi_l^{j,\Gamma}, \Psi_{l'}^{j',\Gamma} \big)_\Gamma = \begin{cases} 1 & \text{if } j = j', l = l', \\ 0 & \text{otherwise.} \end{cases} \tag{5.179}$$

If Φ^Γ in (5.176) is both a u-wavelet frame according to Theorem 5.9 and a basis in $L_2(\Gamma)$ then one has the same situation as in Theorem 1.20. Let again $A^s_{pq}(\Gamma)$ be either $B^s_{pq}(\Gamma)$ or $F^s_{pq}(\Gamma)$ and correspondingly $a^s_{pq}(\mathbb{Z}_\Gamma)$ be either $b^s_{pq}(\mathbb{Z}_\Gamma)$ or $f^s_{pq}(\mathbb{Z}_\Gamma)$.

Proposition 5.32. *Let Γ be a compact n-dimensional C^∞ manifold and let $A^s_{pq}(\Gamma)$ be the spaces as introduced in Definition 5.1. Let Φ^Γ in (5.176) for $u \in \mathbb{N}$ be both a u-wavelet frame according to Theorem 5.9 with the dual system $\big\{ \Psi_l^{j,\Gamma} \big\}$ as above, and a basis in $L_2(\Gamma)$. Let $0 < p \le \infty$ (with $p < \infty$ for the F-spaces), $0 < q \le \infty$, $s \in \mathbb{R}$, and*

$$u > \max(s, \sigma_p - s) \ B\text{-spaces}; \quad u > \max(s, \sigma_{pq} - s) \ F\text{-spaces}. \tag{5.180}$$

Then $f \in D'(\Gamma)$ is an element of $A^s_{pq}(\Gamma)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j^\Gamma} \lambda_l^j \, 2^{-jn/2} \, \Phi_l^{j,\Gamma}, \quad \lambda \in a^s_{pq}(\mathbb{Z}_\Gamma), \tag{5.181}$$

unconditional convergence being in $A^s_{pq}(\Gamma)$ if $p < \infty$, $q < \infty$, and in $A^{s-\varepsilon}_{pq}(\Gamma)$ for any $\varepsilon > 0$ if $p = \infty$ and/or $q = \infty$. The representation (5.181) is unique,

$$\lambda_l^j = \lambda_l^j(f) = 2^{jn/2} \big(f, \Psi_l^{j,\Gamma} \big)_\Gamma \tag{5.182}$$

and

$$I : f \mapsto \big\{ 2^{jn/2} \big(f, \Psi_l^{j,\Gamma} \big)_\Gamma \big\} \tag{5.183}$$

is an isomorphic map of $A^s_{pq}(\Gamma)$ onto $a^s_{pq}(\mathbb{Z}_\Gamma)$. If, in addition, $p < \infty$, $q < \infty$, then Φ^Γ in (5.176) is an unconditional basis in $A^s_{pq}(\Gamma)$.

Proof. The smoothness properties both for Φ^Γ and its dual system $\{\Psi_l^{j,\Gamma}\}$, and also the bi-orthogonality (5.179) are essentially the same as in $\mathbb{R}^n$. Then one obtains the proposition by the same arguments as in the proof of Theorem 1.20 and the references to [T06] given there. □

Remark 5.33. If Φ^Γ in (5.176) is a basis in $L_2(\Gamma)$ then it is a Riesz basis with (5.177), (5.178). But it need not to be an orthogonal basis in contrast to all corresponding wavelet bases in the preceding Chapters 1–3. But the dual system $\{\Psi_l^{j,\Gamma}\}$ has the same smoothness and localisation properties as Φ^Γ. This ensures that Φ^Γ is a basis (isomorphic map) in the same spaces where it is a u-wavelet frame. Examples will be discussed later on.

5.2.2 A criterion

For bounded C^∞ domains in $\mathbb{R}^n$ we obtained in Theorem 5.27 common oscillating u-wavelet frames Φ_r for spaces $A^s_{pq}(\Omega)$ with (5.142). Here $r \in \mathbb{N}$ (excluding now $r = 0$) and $u \in \mathbb{N}$ with $r < u$ are given, where Φ_r comes from (5.155) with Φ^Ω as in (5.147) and $\Phi^{\Gamma,r}$ as in (5.148), (5.153). This shifts the question of whether Φ_r is not only a frame but even a basis from Ω to its boundary $\Gamma = \partial\Omega$. For all admitted k with $k = 0, \ldots, r-1$ we used in (5.148), (5.153) the same boundary system Φ^Γ in (5.150). But this is not necessary. One may choose for each k an individual admitted boundary system Φ^Γ, say Φ^Γ_k. Then one has to replace $\Phi_l^{j-1,\Gamma}(\gamma')$ in (5.153) by $\Phi_{l,k}^{j-1,\Gamma}(\gamma')$ without any changes in the arguments based on (5.148), (5.155). Furthermore one does not need that the system Φ^Γ or, now, the systems Φ^Γ_k, are oscillating on Γ according to Definition 5.5 (ii). These modifications are immaterial for the above considerations, but they will be of some use later on when we apply the following *criterion*.

Proposition 5.34. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ with $n \geq 2$ as in Definition 3.4* (iii). *Let $r \in \mathbb{N}$ and $u \in \mathbb{N}$ with $r < u$. Then the oscillating u-wavelet frame according to Theorem 5.27 and (5.155) is a basis in $A^s_{pq}(\Omega)$ with (5.142) if, and only if, the underlying systems*

$$\Phi^\Gamma_k = \{\Phi^{j,\Gamma}_{l,k} : j \in \mathbb{N}_0;\ l = 1, \ldots, N^\Gamma_j\} \subset C^u(\Gamma) \tag{5.184}$$

on $\Gamma = \partial\Omega$ with $k = 0, \ldots, r-1$ according to Definition 5.5 (i) *and the above explanations are bases in $L_2(\Gamma)$.*

Proof. Step 1. Let Φ^Γ_k in (5.184) be bases in $L_2(\Gamma)$. By Proposition 5.32 they are also bases in all spaces $B^{s-\frac{1}{p}-k}_{pq}(\Gamma)$ of interest, $k = 0, \ldots, r-1$. In order to prove that Φ_r is also a basis it suffices to show that the representation (5.143) is unique. Let

$$0 = \sum_{j=0}^\infty \sum_{l=1}^{N_j} \lambda^j_l\, 2^{-jn/2}\, \Phi^j_l, \quad \lambda \in a^s_{pq}(\mathbb{Z}^\Omega). \tag{5.185}$$

We use the projections $Q^{r,u}$ and $P^{r,u} = \mathrm{id} - Q^{r,u}$ as in Step 3 of the proof of Theorem 5.27. We apply $Q^{r,u}$ to (5.185). By (5.159) and the equality $Q^{r,u} P^{r,u} = Q^{r,u}(\mathrm{id} - Q^{r,u}) = 0$ one obtains as in (5.160)

$$0 = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j^{\Omega}} \lambda_l^{j,\Omega}\, 2^{-jn/2}\, \Phi_l^{j,\Omega}. \tag{5.186}$$

But Φ^{Ω} is a basis. Hence $\lambda_l^{j,\Omega} = 0$. If one applies $\mathrm{tr}_{\Gamma} \frac{\partial^k}{\partial \nu^k}$ with $k = 0, \ldots, r-1$ to (5.185) then one has the k-term in (5.163) and by (5.153) with $\gamma_n = 0$ (in obvious notation) that

$$0 = \sum_{j=1}^{\infty} \sum_{l=1}^{N_j^{\Gamma}} \lambda_{l,k}^{j,\Gamma}\, 2^{-j\frac{n-1}{2}}\, \Phi_{l,k}^{j-1,\Gamma}(\gamma'). \tag{5.187}$$

By assumption, Φ_k^{Γ} is a basis in $L_2(\Gamma)$. Hence $\lambda_{l,k}^{j,\Gamma} = 0$, and finally $\lambda_l^j = 0$ in (5.185). Consequently Φ_r is a basis.

Step 2. Let Φ_r be a basis. We wish to prove that Φ_k^{Γ} in (5.184) are bases in $L_2(\Gamma)$. First we prove that Φ_k^{Γ} with $k = 0$ is a basis in, say, the trace space

$$H^{\varepsilon}(\Gamma) = B^{\varepsilon}_{2,2}(\Gamma) = \mathrm{tr}_{\Gamma}\, B^{\varepsilon+\frac{1}{2}}_{2,2}(\Omega), \quad 0 < \varepsilon < 1. \tag{5.188}$$

Again it is a matter of uniqueness. Let

$$0 = \sum_{j=1}^{\infty} \sum_{l=1}^{N_j^{\Gamma}} \lambda_l^{j,\Gamma}\, 2^{-j\frac{n-1}{2}}\, \Phi_{l,0}^{j-1,\Gamma}(\gamma'), \quad \lambda \in b^{\varepsilon}_{2,2}(\mathbb{Z}_{\Gamma}). \tag{5.189}$$

Extending expansions in $H^{\varepsilon}(\Gamma)$ by (5.152), (5.153) with $k = 0$ to $B^{\varepsilon+\frac{1}{2}}_{2,2}(\Omega)$ one obtains an admitted expansion in $B^{\varepsilon+\frac{1}{2}}_{2,2}(\Omega)$. Since we assume that Φ_r^{Γ} is a basis one has $\lambda_l^{j,\Gamma} = 0$. Hence Φ_0^{Γ} is a basis in $H^{\varepsilon}(\Gamma)$. As a consequence one has the bi-orthogonality (5.179). But now one is in a similar position as in Proposition 5.32 with $H^{\varepsilon}(\Gamma)$ in place of $L_2(\Gamma)$. By the same arguments and references as there it follows that Φ_0^{Γ} is a basis in all admitted spaces and, in particular, in $L_2(\Gamma)$. Similarly for $k = 1, \ldots, r-1$. □

5.2.3 Wavelet bases on intervals and planar domains

Domains in the plane $\mathbb{R}^2$ are called *planar domains*. We deal with wavelet bases on bounded intervals on $\mathbb{R} = \mathbb{R}^1$ and on bounded planar C^{∞} domains in $\mathbb{R}^2$ according to Definition 3.4 (iii). Let $A^s_{pq}(\Omega)$ with $A \in \{B, F\}$ and correspondingly $a^s_{pq}(\mathbb{Z}^{\Omega})$ with $a \in \{b, f\}$ be as introduced in Definitions 2.1 and 5.23.

Theorem 5.35. *Let Ω be either a bounded interval on $\mathbb{R}$ $(n = 1)$ or a bounded planar C^∞ domain in $\mathbb{R}^2$ $(n = 2)$. Let $r \in \mathbb{N}_0$ and $u \in \mathbb{N}$ with $r < u$. Then there is a common oscillating u-wavelet system Φ_r according to Definition 5.25* (ii) *for all p, q, s with*

$$1 \le p < \infty, \quad \frac{1}{p} - 1 < s - r < \frac{1}{p}, \quad \begin{cases} 0 < q < \infty, & B \text{ spaces}, \\ 1 \le q < \infty, & F \text{ spaces}, \end{cases} \tag{5.190}$$

and $A \in \{B, F\}$ such that an element $f \in D'(\Omega)$ belongs to $A^s_{pq}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda^j_l \, 2^{-jn/2} \, \Phi^j_l, \quad \lambda \in a^s_{pq}(\mathbb{Z}^\Omega), \tag{5.191}$$

unconditional convergence being in $A^s_{pq}(\Omega)$. The representation (5.191) is unique,

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda^j_l(f) \, 2^{-jn/2} \, \Phi^j_l, \tag{5.192}$$

where $\lambda^j_l(\cdot) \in A^s_{pq}(\Omega)'$ are linear and continuous functionals on $A^s_{pq}(\Omega)$ and

$$f \mapsto \{\lambda^j_l(f)\} \text{ is an isomorphic map of } A^s_{pq}(\Omega) \text{ onto } a^s_{pq}(\mathbb{Z}^\Omega) \tag{5.193}$$

(u-wavelet basis).

Proof. Step 1. As mentioned in Step 1 of the proof of Theorem 5.27 there is nothing to prove if $r = 0$ and $u \in \mathbb{N}$.

Step 2. Let Ω be a bounded interval, say the unit interval $\Omega = (0, 1)$. Let $r \in \mathbb{N}$, hence $s > 1/p$. In this case, (5.152), (5.153) reduces to

$$g = \operatorname{Ext}^{r-1,u}_\Gamma \{g_0, \ldots, g_{r-1}\} = \chi(x) \sum_{k=0}^{r-1} \frac{x^k}{k!} \, g_k(0) + (1), \tag{5.194}$$

where (1) indicates a corresponding term at $x = 1$. Now one can argue as in the proof of Proposition 5.34.

Step 3. The boundary $\Gamma = \partial\Omega$ of a bounded C^∞ domain in the plane consists of finitely many closed C^∞ curves Γ_j. For each such curve there is a tubular neighbourhood (strip)

$$\Gamma^\varepsilon_j = \big\{ x \in \mathbb{R}^2 : \operatorname{dist}(x, \Gamma_j) < \varepsilon \big\} \quad \text{for some } \varepsilon > 0 \tag{5.195}$$

and a diffeomorphic map ψ_j onto a tubular neighbourhood of

$$\psi_j(\Gamma_j) = \big\{ y \in \mathbb{R}^2 : |y| = 1 \big\} = \mathbb{T}^1, \tag{5.196}$$

interpreting the unit circle as the 1-torus $\mathbb{T}^1$. By Theorem 1.37 there are orthonormal wavelet bases with the desired properties in all spaces $A^s_{pq}(\mathbb{T}^1)$. Both these spaces and also the bases can be transferred by the above diffeomorphic maps ψ_j to Γ_j. Then one has a u-wavelet basis on Γ. Application of Proposition 5.34 proves the theorem in case of planar domains. □

Remark 5.36. The above method works in higher dimensions only in exceptional cases. For example, let Ω be a tube (torus of revolution) in $\mathbb{R}^3$ such that its boundary $\Gamma = \partial\Omega$ can be identified with the 2-torus $\mathbb{T}^2$. Then one has by Theorem 1.37 a wavelet basis on Γ. Application of Proposition 5.34 gives a wavelet basis for the spaces $A^s_{pq}(\Omega)$ with (5.142).

5.3 Wavelet bases: higher dimensions

5.3.1 Introduction

In dimensions one and two we have the satisfactory Theorem 5.35. In higher dimensions, $n \ge 3$, the situation is less favourable. According to Remark 5.36 one has a full counterpart of Theorem 5.35 for the torus of revolution $\mathbb{M}^3$ in $\mathbb{R}^3$ with $\partial\mathbb{M}^3 = \mathbb{T}^2$ (the 2-torus) with the same exceptional values $s - \frac{1}{p} \in \mathbb{N}_0$. In particular it remains open whether the distinguished Sobolev spaces

$$H^s(\mathbb{M}^3) = B^s_{2,2}(\mathbb{M}^3) \quad \text{with } s - \tfrac{1}{2} \in \mathbb{N}_0 \tag{5.197}$$

have a u-wavelet basis. However it comes out that this is not an artefact produced by the method but it lies in the nature of the problem. For example, since $D(\mathbb{M}^3)$ is dense in $H^{1/2}(\mathbb{M}^3)$ it is natural to ask whether there are u-wavelet bases in $H^{1/2}(\mathbb{M}^3)$ with compact supports in $\mathbb{M}^3$. The answer is negative. We return to the problem of non-existence of wavelet bases (and wavelet frames) in Section 6.2.2. Asking for wavelet bases in balls

$$\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\} \tag{5.198}$$

or on spheres

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\} \tag{5.199}$$

the situation is even worse. It comes out that we do not obtain u-wavelet bases by our method in some exceptional spaces including

$$H^s(\mathbb{S}^n) \quad \text{and} \quad H^s(\mathbb{B}^n), \quad n \ge 3, \; s \in \mathbb{N} \text{ and } s - \tfrac{1}{2} \in \mathbb{N}_0. \tag{5.200}$$

In particular it remains open whether the classical Sobolev spaces

$$H^k(\mathbb{S}^n) = W^k_2(\mathbb{S}^n) \quad \text{and} \quad H^k(\mathbb{B}^n) = W^k_2(\mathbb{B}^n), \quad n \ge 3, \; k \in \mathbb{N}, \tag{5.201}$$

have u-wavelet bases. There is a remarkable difference between the exceptional values in (5.197) and in (5.200) with $n = 3$. But it is at least questionable whether this has something to do with the different topology of $\mathbb{M}^3$ and $\mathbb{S}^3$ or $\mathbb{B}^3$. It depends on our method which is based on induction by dimension. As for other methods one may consult the references given at the beginning of Section 5.3.3. We deal in Section 5.3.2 first with spheres and balls. But the method itself works apparently on a much larger scale. It is based on some surgery cutting manifolds and domains in more handsome pieces. The resulting wavelet bases are glued together afterwards. One obtains in this way rather final results for the some classes of domains of the same type as for spheres and balls. But this requires not only some topological considerations (known as the *domain problem*) but also the study of traces and related extensions for some types of non-smooth domains, the finite union of diffeomorphic distortions of cubes. This will be done in Sections 5.3.3, 5.3.4 and in greater detail in Section 6.1.

5.3.2 Wavelet bases on spheres and balls

The sphere $\Gamma = \mathbb{S}^n$ in (5.199) is a compact n-dimensional C^∞ manifold as introduced at the beginning of Section 5.1.2. Let $A^s_{pq}(\Gamma) = A^s_{pq}(\mathbb{S}^n)$ be the spaces according to Definition 5.1. With $\mathbb{Z}_\Gamma$ as in (5.25)–(5.27) the sequence spaces $a^s_{pq}(\mathbb{Z}_\Gamma)$ with $a \in \{b, f\}$ have the same meaning as in Definition 5.7. As for the u-wavelet system $\Phi^\Gamma = \{\Phi^{j,\Gamma}_l\}$ we refer to Definition 5.5 (i) indicating now Γ. In contrast to Theorem 5.9 we do not care any longer whether the corresponding u-wavelet frames are oscillating. The procedure on which we rely now is characterised by some surgery or domain decomposition. In tubular neighbourhoods of the cutting surfaces one cannot guarantee a qualitative cancellation of type (5.32), called oscillation. Although this property remains valid outside these cutting surfaces this may justify to abandon this property. Furthermore, the induction by dimension produces additional exceptional (s, p) values and it is no longer reasonable to ask for common u-wavelet bases in (s, p)-regions. In other words, we ask for u-wavelet bases in a given space $A^s_{pq}(\Gamma)$ and, later on, in $A^s_{pq}(\Omega)$.

Theorem 5.37. *Let $\Gamma = \mathbb{S}^n$ be the n-sphere (5.199) with $n \geq 2$. Let*

$$1 \leq p < \infty, \quad s > \frac{1}{p} - 1, \quad s - \frac{m}{p} \notin \mathbb{N}_0 \text{ for } m = 1, \dots, n-1, \tag{5.202}$$

and $0 < q < \infty$ for the B-spaces, $1 \leq q < \infty$ for the F-spaces. Let $u \in \mathbb{N}$ with $u > s$. Then there is a u-wavelet system

$$\Phi^\Gamma = \{\Phi^{j,\Gamma}_l : j \in \mathbb{N}_0;\ l = 1, \dots, N^\Gamma_j\} \subset C^u(\Gamma) \tag{5.203}$$

according to Definition 5.5 (i) such that $f \in D'(\Gamma)$ belongs to $A^s_{pq}(\Gamma)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N^\Gamma_j} \lambda^j_l \, 2^{-jn/2} \Phi^{j,\Gamma}_l, \quad \lambda \in a^s_{pq}(\mathbb{Z}_\Gamma), \tag{5.204}$$

unconditional convergence being in $A^s_{pq}(\Gamma)$. *The representation* (5.204) *is unique,*

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j^\Gamma} \lambda_l^j(f)\, 2^{-jn/2}\, \Phi_l^{j,\Gamma}, \tag{5.205}$$

where $\lambda_l^j(\cdot) \in A^s_{pq}(\Gamma)'$ *are linear and continuous functionals on* $A^s_{pq}(\Gamma)$ *and*

$$f \mapsto \{\lambda_l^j(f)\} \text{ is an isomorphic map of } A^s_{pq}(\Gamma) \text{ onto } a^s_{pq}(\mathbb{Z}_\Gamma) \tag{5.206}$$

(*u-wavelet basis*).

Proof. Step 1. We decompose $\mathbb{S}^2 = \partial \mathbb{B}^3$ as

$$\mathbb{S}^2 = \mathbb{S}^2_+ \cup \mathbb{S}^2_- \cup \left\{ x \in \mathbb{R}^3 : \; x_1^2 + x_2^2 = 1, \; x_3 = 0 \right\} \tag{5.207}$$

where

$$\mathbb{S}^2_+ = \left\{ x \in \mathbb{R}^3 : \; |x| = 1, \; x_3 > 0 \right\} \tag{5.208}$$

and similarly $\mathbb{S}^2_-$. We contract the northern hemisphere $\mathbb{S}^2_+$ with the North Pole $(0,0,1)$ along the meridians such that the C^∞ dilation is 1 near the North Pole (no contraction) and, say, $1/2$ at the equator. Then one obtains a diffeomorphic map

$$\psi : \mathbb{S}^2_+ \mapsto \left\{ x \in \mathbb{R}^3 : \; |x| = 1, \; 0 < c < x_3 \le 1 \right\} = \mathbb{S}^2_c \tag{5.209}$$

which can be extended to a diffeomorphic map of a tubular neighbourhood of $\mathbb{S}^2_+$ onto a tubular neighbourhood of $\mathbb{S}^2_c$. But $\mathbb{S}^2_c$ can be represented as

$$\mathbb{S}^2_c = \left\{ x \in \mathbb{R}^3 : \; (x_1, x_2) \in B_d, \; x_3 = h(x_1, x_2) \right\} \tag{5.210}$$

with

$$B_d = \left\{ y \in \mathbb{R}^2 : \; |y| < d \right\} \quad \text{for some } d > 0 \text{ and } h \in C^\infty(B_d). \tag{5.211}$$

Any wavelet basis for $\bar{A}^s_{pq}(B_d)$ according to Theorem 3.23 can be transferred to the corresponding spaces $\bar{A}^s_{pq}(\mathbb{S}^2_c)$ and afterwards by the inverse ψ^{-1} of ψ in (5.209) to $\bar{A}^s_{pq}(\mathbb{S}^2_+)$. Similarly for $\bar{A}^s_{pq}(\mathbb{S}^2_-)$. The third set on the right-hand side of (5.207) is a circle, the equator. But now we are in the same situation as in Theorem 5.35 for planar domains with a reference to Proposition 5.34 which in turn relies on the wavelet-friendly extension (5.152), (5.153). But this extension works on both sides of the equator. All other technicalities are covered by Theorem 5.27. This proves the above assertion for $n = 2$.

Step 2. Let $n = 3$. Then

$$\mathbb{S}^3 = \mathbb{S}^3_+ \cup \mathbb{S}^3_- \cup \left\{ x \in \mathbb{R}^4 : \; x_1^2 + x_2^2 + x_3^2 = 1, \; x_4 = 0 \right\} \tag{5.212}$$

is the counterpart of (5.207), (5.208). As in Step 1 one has wavelet bases in $\bar{A}^s_{pq}(\mathbb{S}^3_+)$ and $\bar{A}^s_{pq}(\mathbb{S}^3_-)$. The last set in (5.212) can be identified with $\mathbb{S}^2$. We may assume

$s > 1/p$ (otherwise there is nothing to prove as mentioned in Step 1 of the proof of Theorem 5.35). Then one obtains by Step 1 a wavelet basis in

$$B^{s-\frac{1}{p}}_{pq}(\mathbb{S}^2) \quad \text{if } s - \tfrac{2}{p} = s - \tfrac{1}{p} - \tfrac{1}{p} \notin \mathbb{N}_0. \tag{5.213}$$

In other words for the boundary spaces $B^{s-\frac{1}{p}-k}_{pq}(\mathbb{S}^2)$ underlying Proposition 5.34 and its proof one has the additional restriction (5.213). Hence one needs now not only

$$s - \frac{1}{p} \notin \mathbb{N}_0, \quad \text{but also} \quad s - \frac{2}{p} \notin \mathbb{N}_0. \tag{5.214}$$

But otherwise one can argue as in Step 1 where we now benefit from the possibility that the boundary systems Φ^{Γ}_k may be different for different values of k. This proves the theorem in case of $n = 3$. The rest is now a matter of induction. □

However the arguments on which the extension from $\mathbb{S}^2$ to $\mathbb{S}^3$ in (5.212), (5.213) relies can also be used for the step from $\mathbb{S}^2 = \partial\mathbb{B}^3$ to $\mathbb{B}^3$. By iteration one has the counterpart of Theorem 5.37 for $\mathbb{B}^n$ with $n \ge 3$ in place of $\mathbb{S}^n$. (In case of $n = 2$ one has Theorem 5.35.) Although everything is largely parallel to the above theorem it seems to be reasonable to give an explicit formulation switching to a slightly more general situation. The boundary $\Gamma = \partial\Omega$ of a bounded C^∞ domain in $\mathbb{R}^n$ consists of finitely many connected components. We say that such a component is diffeomorphic to $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$ if there is a diffeomorphic map of a tubular neighbourhood of this component onto a corresponding tubular neighbourhood of $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$. Let $A^s_{pq}(\Omega)$ with $A \in \{B, F\}$ and correspondingly $a^s_{pq}(\mathbb{Z}^\Omega)$ with $a \in \{b, f\}$ be as in Definitions 2.1 and 5.23. Equivalences $\sim$ must be understood as in (3.4), (3.5).

Theorem 5.38. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ with $n \ge 3$ according to Definition 3.4* (iii) *such that each of its connected boundary components is diffeomorphic to $\mathbb{S}^{n-1}$ as explained above. Let*

$$1 \le p < \infty, \quad s > \frac{1}{p} - 1, \quad s - \frac{m}{p} \notin \mathbb{N}_0 \text{ for } m = 1, \dots, n-1, \tag{5.215}$$

and $0 < q < \infty$ for the B-spaces, $1 \le q < \infty$ for the F-spaces. Let $u \in \mathbb{N}$ with $u > s$. Then there is an oscillating u-wavelet system

$$\Phi^\Omega = \left\{\Phi^{j,\Omega}_l : j \in \mathbb{N}_0 : l = 1, \dots, N^\Omega_j\right\} \subset C^u(\Omega), \quad N^\Omega_j \in \mathbb{N}, \tag{5.216}$$

(where $N^\Omega_j \sim 2^{jn}$) in $\bar{\Omega}$ according to Definition 5.25 (ii) *such that $f \in D'(\Omega)$ belongs to $A^s_{pq}(\Omega)$ if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N^\Omega_j} \lambda^j_l \, 2^{-jn/2} \, \Phi^{j,\Omega}_l, \quad \lambda \in a^s_{pq}(\mathbb{Z}^\Omega), \tag{5.217}$$

unconditional convergence being in $A^s_{pq}(\Omega)$. *The representation* (5.217) *is unique,*

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j^\Omega} \lambda_l^j(f)\, 2^{-jn/2}\, \Phi_l^{j,\Omega}, \tag{5.218}$$

where $\lambda_l^j(\cdot) \in A^s_{pq}(\Omega)'$ *are linear and continuous functionals on* $A^s_{pq}(\Omega)$ *and*

$$f \mapsto \{\lambda_l^j(f)\} \text{ is an isomorphic map of } A^s_{pq}(\Omega) \text{ onto } a^s_{pq}(\mathbb{Z}^\Omega) \tag{5.219}$$

(u*-wavelet basis*).

Proof. Let $\Omega = \mathbb{B}^n$ with $n \geq 3$. As mentioned in Step 1 of the proof of Theorem 5.27 there is nothing to prove if $s < 1/p$. Let $s > 1/p$. Then it follows from (5.202) that one can apply Theorem 5.37 to the needed boundary spaces $B^{s-\frac{1}{p}-k}_{pq}(\mathbb{S}^{n-1})$ with $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$. Now one can use Proposition 5.34 (in its full generality) as in Step 2 of the proof of Theorem 5.37. Let $\partial\Omega$ be diffeomorphic to $\mathbb{S}^{n-1}$ as explaind above. But diffeomorphic distortions of the boundary wavelets in Definition 5.25 (i) result in wavelets of the same type. Then one obtains again the desired assertion. This can be extended to boundaries consisting of several components which are diffeomorphic to $\mathbb{S}^{n-1}$. □

Remark 5.39. We complained in Introduction 5.3.1 that some of the most interesting function spaces on spheres and balls are not covered by the above Theorems 5.37, 5.38. The situation is better for the distinguished wavelet frames according to Theorem 5.27. But even there remain the exceptional cases with $s - \frac{1}{p} \in \mathbb{N}_0$. One can avoid any exceptional cases if one restricts the wavelet bases on $\mathbb{R}^n$ according to Theorem 1.20 to Ω. We return later on in Section 5.4 to this point. But the resulting frames cannot be converted into u-wavelet bases. There is no counterpart of Proposition 5.34.

5.3.3 Wavelet bases in cellular domains and manifolds

For bounded C^∞ domains in $\mathbb{R}^n$ we have satisfactory frame representations for the spaces $A^s_{pq}(\Omega)$ covered by Theorem 5.27. Related assertions for bases are more difficult, especially in higher dimensions, which means $n \geq 3$. In contrast to $n = 2$ where one has Theorem 5.35, the related higher dimensional counterpart in Theorem 5.38 applies only to very special domains. What about bounded domains in $\mathbb{R}^n$ with $n \geq 3$ where the connected boundary components are not diffeomorphic to $\mathbb{S}^{n-1}$ or $\mathbb{T}^{n-1}$? Basically one can try to decompose a domain into simpler standard domains, to deal with bases in these standard domains, and to glue together the outcome. This is the so-called *domain problem* which attracted a lot of attention. But especially the question how to glue together bases in standard domains is a rather tricky business. The most impressive contributions to these problems are still the papers by Z. Ciesielski and T. Figiel, [CiF83a], [CiF83b], [Cie84] dealing with common spline bases in

$C^k(M)$, where $k \in \mathbb{N}$,

the classical Sobolev spaces $W_p^k(M)$, where $k \in \mathbb{N}$, $1 \le p < \infty$, and

the classical Besov spaces $B_{pq}^s(M)$, where $1 \le p < \infty$, $1 \le q < \infty$, $s \in \mathbb{R}$,

on compact C^∞ manifolds M (with and without boundary). The final goal is reached after more than 100 pages in [Cie84], Theorem C. There are several proposals in literature to employ and to modify this approach in the context of wavelet bases. We refer to [Dah97], Section 10, [Dah01], Section 9, [Coh03], p. 130, with a reference to the original paper [DaS99], and [HaS04], [JoK07].

We have no final answers. But we wish to look at problems of this type in the context of our approach. Recall what is meant by a *polyhedron* (polyhedral domain) in $\mathbb{R}^n$. In $\mathbb{R} = \mathbb{R}^1$ open bounded intervals are called polyhedrons. In $\mathbb{R}^n$ with $n \ge 2$ a set ω is said to be a polyhedron if it is a bounded simply connected Lipschitz domain such that its boundary $\partial\omega$ consists of finitely many faces which are $(n-1)$-dimensional polyhedrons in $(n-1)$-dimensional hyper-planes in $\mathbb{R}^n$. We do not discuss geometrical and topological aspects of this definition. Angles resulting from the intersection of $(n-1)$-dimensional faces are usually assumed to be non-zero. But for our purpose this is neither necessary nor suitable (one may think about domain decompositions as for $\mathbb{S}^3$ in (5.212)). However we are more interested in the analytic aspect with the unit cube

$$Q = \{x \in \mathbb{R}^n : 0 < x_r < 1;\ r = 1, \ldots, n\}$$

as the proto-type of a polyhedron. In Definition 5.17 we introduced the spaces $C^\infty(\Omega)$ in domains Ω in $\mathbb{R}^n$. Recall that a one-to-one *map* ψ from a bounded domain Ω in $\mathbb{R}^n$ onto a bounded domain ω in $\mathbb{R}^n$,

$$\psi : \Omega \ni x \mapsto y = \psi(x) \in \omega,$$

is called a *diffeomorphic map* if

$$\psi_l \in C^\infty(\Omega) \quad \text{and} \quad (\psi^{-1})_l \in C^\infty(\omega); \quad l = 1, \ldots, n,$$

for the components ψ_l of ψ and $(\psi^{-1})_l$ of its inverse ψ^{-1},

$$\psi^{-1} \circ \psi = \text{id in } \Omega \quad \text{and} \quad \psi \circ \psi^{-1} = \text{id in } \omega.$$

A bounded domain Ω in $\mathbb{R}^n$ is said to be diffeomorphic to a bounded domain ω in $\mathbb{R}^n$ if there is a diffeomorphic map ψ of a neighbourhood of $\overline{\Omega}$ onto a neighbourhood of $\overline{\omega}$ with $\omega = \psi(\Omega)$. If Γ is a set in $\mathbb{R}^n$ then Γ° is the largest open set in $\mathbb{R}^n$ with $\Gamma^\circ \subset \Gamma$ (the interior of Γ).

Definition 5.40. (i) A domain Ω in $\mathbb{R}^n$, $n \ge 2$, is said to be cellular if it is a bounded Lipschitz domain according to Definition 3.4 (iii) which can be represented as

$$\Omega = \Big(\bigcup_{l=1}^{L} \overline{\Omega_l} \Big)^\circ \quad \text{with } \Omega_l \cap \Omega_{l'} = \emptyset \text{ if } l \neq l', \tag{5.220}$$

such that each Ω_l is diffeomorphic to a polyhedron.

(ii) A compact n-dimensional C^∞ manifold Γ as introduced at the beginning of Section 5.1.2, furnished with an atlas $\{V_m, \psi_m\}_{m=1}^M$, is said to be cellular if there are pairwise disjoint open subsets Γ_j of Γ with

$$\Gamma = \bigcup_{m=1}^{M} \overline{\Gamma_m}, \quad \text{and} \quad \overline{\Gamma_m} \subset V_m, \tag{5.221}$$

such that each $\psi_m(\Gamma_m)$ is a cellular domain, $\psi_m(\overline{\Gamma_m}) \subset U_m \subset \mathbb{R}^n$ if $n \geq 2$ or a bounded interval if $n = 1$.

Remark 5.41. Any bounded C^∞ domain in $\mathbb{R}^n$ according to Definition 3.4 (iii) is cellular. Based on the above references one can presumably show that

any compact C^∞ manifold is cellular.

But we did not check this assertion. We are interested here only on $\Gamma = \partial\Omega$, where Ω is a bounded C^∞ domain.

Proposition 5.42. *The boundary $\Gamma = \partial\Omega$ of a bounded C^∞ domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition* 3.4 (iii) *is an $(n-1)$-dimensional cellular compact manifold.*

Proof. Let $\{V_m, \psi_m\}_{m=1}^M$ be an atlas of Γ originating from the special maps according to Definition 3.4 (ii,iii) with $U_m = \psi_m(V_m) \subset \mathbb{R}^{n-1}$. We choose a suitable cellular domain W_1 in U_1 with $\overline{W_1} \subset U_1$ and $\Gamma_1 = \psi_1^{-1}(W_1)$ such that $\partial\Gamma_1 \subset \bigcup_{m=2}^M V_m$. Then application of ψ_2 to $V_2 \setminus \Gamma_1$ produces in U_2 a boundary which fits in the scheme (assuming that V_2 and Γ_1 have a non-empty intersection). Now one can construct a cellular domain W_2 in $\psi_2(V_2 \setminus \Gamma_1)$ with a boundary $\partial\Gamma_2$ of $\Gamma_2 = \psi_2^{-1}(W_2)$ covering parts of $\partial(V_2 \setminus \Gamma_1)$ such that the remaining parts of $\partial(V_2 \setminus \Gamma_1)$ and of $\partial\Gamma_2$ are contained in $\bigcup_{m=3}^M V_m$. Iteration gives the desired assertion. □

Recall that the characteristic function χ_+ of $\mathbb{R}^n_+$ is a pointwise multiplier for

$$A^s_{pq}(\mathbb{R}^n), \quad s \in \mathbb{R}, \quad 0 < p \leq \infty, \quad 0 < q \leq \infty, \tag{5.222}$$

$(p < \infty$ for the F-spaces) if, and only if,

$$\max\left(n\left(\frac{1}{p} - 1\right), \frac{1}{p} - 1\right) < s < \frac{1}{p}. \tag{5.223}$$

A proof may be found in [RuS96], Section 4.6.3, p. 208. We refer for the history of this substantial assertion to [RuS96], p. 258. This assertion extends to cubes, polyhedrons, their diffeomorphic images and hence also to cellular domains according to Definition 5.40 (i). Since cellular domains are bounded Lipschitz domains one obtains by Corollary 3.25 that one can apply Theorem 3.23. We complement now this assertion. Let $A^s_{pq}(\Omega)$ with $A \in \{B, F\}$ be as before and $a^s_{pq}(\mathbb{Z}_\Omega)$ with $a \in \{b, f\}$ as used in

Theorem 3.23 with a reference to Definition 2.6. We stick at orthonormal u-wavelet bases in $L_2(\Omega)$ as the starting point, but we do not care any longer about optimal values of u and, as a consequence, common bases in some (s, p)-regions. This can be done following the proofs but the outcome is no longer very convincing.

Theorem 5.43. *Let Ω be a cellular domain in $\mathbb{R}^n$ according to Definition 5.40 (i). Let for $u \in \mathbb{N}$,*

$$\left\{\Phi_l^j : j \in \mathbb{N}_0;\ l = 1, \dots, N_j\right\} \tag{5.224}$$

be an orthonormal u-wavelet basis in $L_2(\Omega)$ according to Theorem 2.33 and Definitions 2.31, 2.4. Let $A^s_{pq}(\Omega)$ be the spaces as introduced in Definition 2.1. Let $0 < p \le \infty$ (with $p < \infty$ for the F-spaces), $0 < q \le \infty$, and

$$-\infty < s < \min\left(\frac{1}{p}, \frac{n}{n-1}\right) \tag{5.225}$$

(which means $s < 1/p$ if $n = 1$). Let $u > u(A^s_{pq})$ where $u(B^s_{pq}) = u(s,p)$ and $u(F^s_{pq}) = u(s,p,q)$ are sufficiently large natural numbers. Then $f \in D'(\Omega)$ is an element of $A^s_{pq}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda_l^j \, 2^{-jn/2} \, \Phi_l^j, \quad \lambda \in a^s_{pq}(\mathbb{Z}_\Omega), \tag{5.226}$$

unconditional convergence being in $D'(\Omega)$ and in any space $A^\sigma_{pq}(\Omega)$ with $\sigma < s$. Furthermore, if $f \in A^s_{pq}(\Omega)$ then the representation (5.226) is unique where $\lambda_l^j(\cdot) \in A^s_{pq}(\Omega)'$ are linear and continuous functionals on $A^s_{pq}(\Omega)$ and

$$f \mapsto \left\{\lambda_l^j(f)\right\} \text{ is an isomorphic map of } A^s_{pq}(\Omega) \text{ onto } a^s_{pq}(\mathbb{Z}_\Omega) \tag{5.227}$$

(u-wavelet basis).

Proof. We apply Theorem 3.23 to the upper lines in (3.45), (3.46) with $p < \infty$ and $q \ge \min(1, p)$ in case of F-spaces, hence

$$\tilde{A}^s_{pq}(\Omega), \quad 0 < p < \infty,\ 0 < q \le \infty,\ s > \sigma_p = n\left(\tfrac{1}{p} - 1\right)_+. \tag{5.228}$$

Furthermore the characteristic function χ_Ω of Ω is a pointwise multiplier in the corresponding spaces $A^s_{pq}(\mathbb{R}^n)$ if, in addition, $s < 1/p$. This requires $\frac{1}{p} < \frac{n}{n-1}$. One has by Definition 2.1 that

$$\tilde{A}^s_{pq}(\Omega) = A^s_{pq}(\Omega) \quad \text{if } 0 < \tfrac{1}{p} < \tfrac{n}{n-1},\ \sigma_p < s < \tfrac{1}{p}. \tag{5.229}$$

A cellular domain is in particular a bounded Lipschitz domain. Hence one can apply the extended complex interpolation method according to Proposition 4.15 and Remark 4.16. One obtains

$$F^s_{pq}(\Omega) = \left[F^{s_0}_{p_0 q_0}(\Omega), F^{s_1}_{p_1 q_1}(\Omega)\right]_\theta, \quad 0 < \theta < 1, \tag{5.230}$$

with

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{5.231}$$

If $0 < p < \infty$, $0 < q \le \infty$ and s as in (5.225) then one finds spaces $F^{s_0}_{p_0 q_0}(\Omega)$ which fit in (5.229), $q_0 \ge \min(1, p_0)$ and

$$F^{s_1}_{p_1 q_1}(\Omega), \quad 0 < p_1 < \infty,\ 0 < q_1 \le \infty,\ s_1 < 0, \tag{5.232}$$

covered by the third line in (3.45) with (5.230), (5.231). We apply Theorem 3.23 to $F^{s_0}_{p_0 q_0}(\Omega)$ and $F^{s_1}_{p_1 q_1}(\Omega)$ with the same underlying u-wavelet basis $\{\Phi^j_l\}$ in $L_2(\Omega)$. On the sequence side one has

$$f^s_{pq}(\mathbb{Z}_\Omega) = \big[f^{s_0}_{p_0 q_0}(\mathbb{Z}_\Omega), f^{s_1}_{p_1 q_1}(\mathbb{Z}_\Omega) \big]_\theta. \tag{5.233}$$

This follows from the comments and references in the proof of Proposition 3.21. Then the interpolation property for the extended complex method ensures that the isomorphic map (3.104), (3.105) between the corner spaces in (5.230), (5.233) generates also an isomorphic map between $F^s_{pq}(\Omega)$ and $f^s_{pq}(\mathbb{Z}_\Omega)$. Similarly one can argue for the B-spaces. □

By Theorem 5.9 we have wavelet frames for all spaces $A^s_{pq}(\Gamma)$ on compact n-dimensional C^∞ manifolds. It seems to be a tricky business to convert these wavelet frames into wavelet bases. For n-spheres $\mathbb{S}^n$ one has Theorem 5.37 which in turn resulted in Theorem 5.38. Now we complement these assertions as follows. Let $a^s_{pq}(\mathbb{Z}_\Gamma)$ be the sequence spaces according to Definition 5.7.

Corollary 5.44. *Let Γ be a compact n-dimensional cellular C^∞ manifold according to Definition 5.40* (ii) *and let $A^s_{pq}(\Gamma)$ be the spaces as introduced in Definition 5.1. Let*

$$0 < p < \infty, \quad 0 < q < \infty, \quad \max\Big(n\Big(\frac{1}{p} - 1\Big), \frac{1}{p} - 1 \Big) < s < \frac{1}{p}. \tag{5.234}$$

Then there is a u-wavelet system $\{\Phi^j_l\}$ according to Definition 5.5 (i) *such that $f \in D'(\Gamma)$ is an element of $A^s_{pq}(\Gamma)$ if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda^j_l \, 2^{-jn/2} \, \Phi^j_l, \quad \lambda \in a^s_{pq}(\mathbb{Z}_\Gamma), \tag{5.235}$$

unconditional convergence being in $A^s_{pq}(\Gamma)$. Furthermore, if $f \in A^s_{pq}(\Gamma)$ then the representation (5.235) *is unique where $\lambda^j_l(\cdot) \in A^s_{pq}(\Gamma)'$ are linear and continuous functionals on $A^s_{pq}(\Gamma)$ and*

$$f \mapsto \{\lambda^j_l(f)\} \text{ is an isomorphic map of } A^s_{pq}(\Gamma) \text{ onto } a^s_{pq}(\mathbb{Z}_\Gamma) \tag{5.236}$$

(u-wavelet basis).

Proof. Let Γ be decomposed according to Definition 5.40 (ii) with (5.221). Then $\psi_m(\Gamma_m)$ is cellular in $\mathbb{R}^n$. With p, q, s as in (5.234) the characteristic function χ^m of $\psi_m(\Gamma_m)$ is a pointwise multiplier in $A^s_{pq}(\mathbb{R}^n)$ with (5.222), (5.223). Now one can apply Theorem 5.43 to

$$\chi^m(\cdot)\,\big[(\chi_m f)\circ\psi_m^{-1}\big](\cdot) \quad \text{in } \psi_m(\Gamma_m),$$

in the notation of Definition 5.1. Clipping together the M charts one obtains the desired assertion. □

Remark 5.45. We proved a little bit more than stated. The u-wavelet basis $\{\Phi^j_l\}$ in Theorem 5.43 comes from Theorem 3.23. Then one has by (3.104) that

$$\lambda^j_l(g) = 2^{jn/2}(g, \Phi^j_l). \tag{5.237}$$

This can be transferred to the above cellular compact C^∞ manifold as

$$\lambda^j_l(f) = 2^{jn/2}\,(f, \Psi^j_l)_\Gamma, \tag{5.238}$$

where $\{\Psi^j_l\}$ is a suitable u-wavelet system according to Definition 5.5 (i).

5.3.4 Wavelet bases in C^∞ domains and cellular domains

We return to the problems discussed at the beginning of Section 5.3.3 asking for wavelet bases in bounded C^∞ domains not covered by Theorem 5.38. The assertions obtained there hint how to proceed.

Proposition 5.46. *Theorem 5.38 can be extended to spaces $A^s_{pq}(\Omega)$ with*

$$1 \le p < \infty, \quad \frac{1}{p} - 1 < s < \frac{2}{p}, \quad s \ne \frac{1}{p}, \tag{5.239}$$

q as there, in (arbitrary) bounded C^∞ domains in $\mathbb{R}^n$, $n \ge 3$.

Proof. As mentioned in Step 1 of the proof of Theorem 5.27 there is nothing to prove if $s < 1/p$. Let $1/p < s < 2/p$. As in the proof of Theorem 5.38 one needs now a wavelet basis for the boundary space $B^{s-\frac{1}{p}}_{pq}(\Gamma)$ with $\Gamma = \partial\Omega$. It follows from Proposition 5.42 that Γ is an $(n-1)$-dimensional cellular manifold. Then one obtains the desired wavelet basis from Corollary 5.44. □

Theorem 5.47. *Theorem 5.38 with*

$$1 \le p < \infty, \quad s > \frac{1}{p} - 1, \quad s - \frac{m}{p} \notin \mathbb{N}_0 \ \textit{for } m = 1, \dots, n, \tag{5.240}$$

instead of (5.215), remains valid for any bounded C^∞ domain in $\mathbb{R}^n$, $n \ge 3$, according to Definition 3.4 (iii).

Remark 5.48. Theorem 5.35 covers the dimensions one and two. In case of $s < 2/p$ and $n \geq 3$ we have Proposition 5.46. Its proof is based on the observation that $\Gamma = \partial\Omega$ is a cellular manifold and that it is sufficient to construct wavelet bases inside each cell separately. Boundary values do not play any role. If $s > 2/p$ then the situation is different. We outline now how to proceed and return in detail to this problem in Section 6.1 below. By Proposition 5.34 and its proof we must find wavelet bases in

$$B^{s-\frac{1}{p}-k}_{pq}(\Gamma) \quad \text{for } k \in \mathbb{N}_0 \text{ with } s - \tfrac{1}{p} - k > 0$$

on the cellular C^∞ manifold $\Gamma = \partial\Omega$. This reduces the problem to find wavelet bases for

$$B^s_{pq}(Q), \quad 1 \leq p < \infty,\ 0 < q \leq \infty,\ s > 2/p, \tag{5.241}$$

where Q is a cube in $\mathbb{R}^n$ (switching from $n-1$ to n), say,

$$Q = \{x \in \mathbb{R}^n : 0 < x_l < 1\} \tag{5.242}$$

as a proto-type of a polyhedron. The worst case (for our task) is $s > n/p$. Then $f \in B^s_{pq}(Q)$ has boundary values in the corner points of Q. Along the edges, say, $I_1 = \{x \in \mathbb{R}^n : 0 < x_1 < 1,\ x' = 0\}$ one can construct wavelet bases for $B^{s-\frac{n-1}{p}}_{pq}(I_1)$ as in Step 2 of the proof of Theorem 5.35 if $s - \frac{n-1}{p} - \frac{1}{p} \not\in \mathbb{N}_0$. This gives a wavelet basis on all edges of, say, a square Q in (5.242) with $n = 2$. The wavelet-friendly extension operator as used in Theorem 5.14 fits pretty well in this scheme. Hence there is a good chance to obtain wavelet bases in $B^{s-\frac{n-2}{p}}_{pq}(Q)$ for the square Q where one has again to exclude spaces with $s - \frac{l}{p} \in \mathbb{N}_0$ where $l = n, n-1$. This requires a closer look at what happens near the corner points. Now one can proceed in this way shifting wavelet bases on 2-dimensional faces of a 3-dimensional cube to this cube. Then one can prove the above assertions by iteration. It requires some care what happens in vertices, edges, faces etc. This is a somewhat tricky business. Details are shifted to Section 6.1. These arguments apply equally to cellular manifolds and cellular domains. Hence Theorem 5.47 can be complemented as follows.

Theorem 5.49. *Theorem* 5.38 *with* (5.240) *in place of* (5.215) *remains valid for any cellular domain in* $\mathbb{R}^n$, $n \geq 2$, *according to Definition* 5.40 (i).

Remark 5.50. As mentioned above, details are shifted to Section 6.1. Both Theorems 5.35, 5.38 and Theorems 5.47, 5.49 apply to spaces $A^s_{pq}(\Omega)$ satisfying at least

$$1 \leq p < \infty, \quad s > \frac{1}{p} - 1, \quad s - \frac{1}{p} \not\in \mathbb{N}_0, \tag{5.243}$$

and $0 < q < \infty$ for the B-spaces, $1 \leq q < \infty$ for the F-spaces. The considerations in Section 6.2.2 show that the restriction $s - \frac{1}{p} \not\in \mathbb{N}_0$ is natural. By Theorem 5.35 there are no further restrictions if Ω is a bounded interval ($n = 1$) or a bounded planar C^∞ domain ($n = 2$). If the bounded planar domain is only cellular (for example the square Q in (5.242), $n = 2$) then Theorem 5.49 and (5.240) also exclude $s - \frac{2}{p} \in \mathbb{N}_0$. In higher dimensions, $n \geq 3$, Theorems 5.47, 5.49 exclude $s - \frac{m}{p} \in \mathbb{N}_0$ for $m = 1, \ldots, n$. Hence Theorem 5.38 is a little bit better and as described at the beginning of Section 5.3.1 for the torus of revolution (or tube) $\mathbb{M}^3$ in $\mathbb{R}^3$ the situation is much better. But we do not know whether there is really a difference between smooth domains and cellular domains or whether even the topology of $\Gamma = \partial\Omega$ plays a role. But it is clear that the method *induction by dimension* as used in the proof of Theorem 5.38 and also in connection with Theorems 5.47, 5.49 does not work in these exceptional cases. Let, for example,

$$\Omega = Q = \big\{x = (x_1, x_2) \in \mathbb{R}^2,\ 0 < x_1 < 1,\ 0 < x_2 < 1\big\} \tag{5.244}$$

be the unit square in the plane $\mathbb{R}^2$. Then $H^1(Q) = W_2^1(Q) = B^1_{2,2}(Q)$ is an exceptional space. Let $\Gamma = \partial\Omega = \bigcup_{l=1}^4 I_l$, where I_l are the four sides of Q,

$$I_1 = \{(t, 0) :\ 0 \le t \le 1\}, \quad I_2 = \{(0, t) :\ 0 \le t \le 1\}, \tag{5.245}$$

and similarly I_3, I_4. Then the trace space $\mathrm{tr}_\Gamma\, H^1(Q)$ is the collection of tuples of the form $g = (g_1, g_2, g_3, g_4)$ with

$$g_l \in H^{1/2}(I_l), \quad l = 1, \ldots, 4, \tag{5.246}$$

such that in addition

$$\int_0^{1/2} \frac{|g_1(t) - g_2(t)|^2}{t}\, \mathrm{d}t + ++ < \infty \tag{5.247}$$

where $+++$ indicates similar integrals at the other 3 corner points. We refer to [Gri85], Theorem 1.5.2.3, 1.5.2.4, pp. 43, 47, and [Gri92], Theorem 1.4.6, p. 17, where one finds also corresponding assertions for some spaces B^s_{pq} in polygonal domains in $\mathbb{R}^2$. The coupling (5.247) of the boundary values at different edges does not fit in our approach. This applies also to $s - \frac{m}{p} \in \mathbb{N}_0$ excluded in (5.240).

5.4 Wavelet frames, revisited

5.4.1 Wavelet frames in Lipschitz domains

For spaces $A^s_{pq}(\Omega)$ having boundary values at $\Gamma = \partial\Omega$ we constructed in Theorem 5.27 oscillating wavelet frames adapted to the underlying bounded C^∞ domains Ω. This paved the way to obtain afterwards at least in some cases wavelet bases. If one does not care about having oscillations near the boundary one may ask what happens if one restricts wavelet bases on $\mathbb{R}^n$ according to Theorem 1.20 to domains Ω hoping that one obtains at least frames. As before we use $A^s_{pq}(\Omega)$ with $A \in \{B, F\}$ and correspondingly $a^s_{pq}(\mathbb{Z}^\Omega)$ with $a \in \{b, f\}$ according to Definitions 2.1 and 5.23. Equivalences $\sim$ must be understood as in (3.4), (3.5). As explained in Remark 5.11 a system $\{\Phi_l^j\}$ is called a *frame* if it is *stable* (with respect to a natural sequence space) and *optimal*. Let σ_p and σ_{pq} be the same numbers as in (1.32).

Theorem 5.51. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition 3.4* (iii). *For any $u \in \mathbb{N}$ there is a u-wavelet system $\{\Phi_l^j\}$ according to Definition 5.25* (i) *with the following properties.*

(i) *Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, and*

$$u > \max(s, \sigma_p - s). \tag{5.248}$$

Then $f \in D'(\Omega)$ is an element of $B^s_{pq}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda_l^j \, 2^{-jn/2} \, \Phi_l^j, \quad \lambda \in b^s_{pq}(\mathbb{Z}^\Omega), \tag{5.249}$$

unconditional convergence being in $B^s_{pq}(\Omega)$ if $p < \infty$, $q < \infty$, and in $B^{s-\varepsilon}_{pq}(\Omega)$ for any $\varepsilon > 0$ if $p = \infty$ and/or $q = \infty$. Furthermore,

$$\|f \,|B^s_{pq}(\Omega)\| \sim \inf \|\lambda \,|b^s_{pq}(\mathbb{Z}^\Omega)\|, \tag{5.250}$$

where the infimum is taken over all admissible representations (5.249) (*equivalent quasi-norms*). *Any $f \in B^s_{pq}(\Omega)$ can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda_l^j(f) \, 2^{-jn/2} \, \Phi_l^j \tag{5.251}$$

where $\lambda_l^j(\cdot) \in B^s_{pq}(\Omega)'$ are linear and continuous functionals on $B^s_{pq}(\Omega)$ and

$$\|f \,|B^s_{pq}(\Omega)\| \sim \|\lambda(f) \,|b^s_{pq}(\mathbb{Z}^\Omega)\| \tag{5.252}$$

(*u-wavelet frame*).

(ii) *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, and*

$$u > \max(s, \sigma_{pq} - s). \tag{5.253}$$

Then $f \in D'(\Omega)$ is an element of $F^s_{pq}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda_l^j \, 2^{-jn/2} \, \Phi_l^j, \quad \lambda \in f^s_{pq}(\mathbb{Z}^\Omega), \tag{5.254}$$

unconditional convergence being in $F^s_{pq}(\Omega)$ if $q < \infty$ and in $F^{s-\varepsilon}_{pq}(\Omega)$ for any $\varepsilon > 0$ if $q = \infty$. Furthermore,

$$\|f \,|F^s_{pq}(\Omega)\| \sim \inf \|\lambda \,|f^s_{pq}(\mathbb{Z}^\Omega)\| \tag{5.255}$$

where the infimum is taken over all admissible representations (5.254). *Any* $f \in F^s_{pq}(\Omega)$ *can be represented as* (5.251) *where* $\lambda^j_l(\cdot) \in F^s_{pq}(\Omega)'$ *are linear and continuous functionals on* $F^s_{pq}(\Omega)$ *and*

$$\|f\,|F^s_{pq}(\Omega)\| \sim \|\lambda(f)\,|f^s_{pq}(\mathbb{Z}^\Omega)\| \tag{5.256}$$

(*u-wavelet frame*).

Proof. Let $\{\Psi^j_{G,m}\}$ be the same orthonormal wavelet bases in $L_2(\mathbb{R}^n)$ as used in Theorem 1.20 based on (1.87)–(1.92). If the support of $\Psi^j_{G,m}$ has a non-empty intersection with Ω then its restriction to Ω is denoted as Φ^j_l,

$$\Phi^j_l(x) = \Psi^j_{G,m}(x), \quad x \in \bar{\Omega},\ l = l(G,m), \tag{5.257}$$

suitably re-numbered. Lipschitz domains are regular domains as described in Remark 4.29 with references to [ET96], Section 2.5, and [TrW96]. Then

$$2^{-j(s-\frac{n}{p})}\,2^{-jn/2}\,\Phi^j_l = a_{jl} \tag{5.258}$$

are Ω-(s,p)-atoms. If $f \in D'(\Omega)$ is represented by (5.249) or (5.254) then $f \in A^s_{pq}(\Omega)$ and

$$\|f\,|A^s_{pq}(\Omega)\| \le c\,\|\lambda\,|a^s_{pq}(\mathbb{Z}^\Omega)\|. \tag{5.259}$$

We prove the converse. As mentioned in Corollary 4.12 (i) with a reference to [Ry99] there is a universal linear and bounded extension operator

$$\mathrm{ext}\colon\ A^s_{pq}(\Omega) \hookrightarrow A^s_{pq}(\mathbb{R}^n). \tag{5.260}$$

With $f \in A^s_{pq}(\Omega)$ one has by Theorem 1.20 that

$$\mathrm{ext}\,f = \sum_{j,G,m} \lambda^{j,G}_m\,2^{-jn/2}\,\Psi^j_{G,m}, \quad \lambda^{j,G}_m = 2^{jn/2}\left(\mathrm{ext}\,f, \Psi^j_{G,m}\right). \tag{5.261}$$

The restriction to Ω gives the desired result. □

Remark 5.52. The proof relies on the existence of (interior and boundary) atoms and extension operators. But this applies also to other cases. It will be the subject of the next Section 5.4.2. We add a discussion and compare the above theorem with other assertions obtained so far. If supp $\Psi^j_{G,m} \subset \Omega$ in (5.257) then the resulting wavelets Φ^j_l preserve all the nice properties of the $\mathbb{R}^n$-wavelets. The situation is less favourable for the wavelets with supports intersecting $\Gamma = \partial\Omega$. They need not to be linearly independent any longer. The multiresolution property is inherited by Ω from $\mathbb{R}^n$ and there is a temptation to apply the standard orthonormalisation procedure to the boundary wavelets. But this might be rather unstable. There is no control about the resulting constants and also the desired localisation cannot be guaranteed. Furthermore, although the proof of the extension operator used in (5.261) is constructive (via local means) it is rather involved. In other words, to calculate for given $f \in A^s_{pq}(\Omega)$ the optimal coefficients $\lambda^j_l(f)$ seems to be rather tricky. It might be a better choice, also from the point of numerical calculations, to rely on the constrained wavelet expansions in Lipschitz domains according to Theorem 4.23. In Theorem 5.27 we described wavelet frames for spaces $A^s_{pq}(\Omega)$ in bounded C^∞ domains under additional restrictions for the parameters p, q, s. The related boundary wavelets are oscillating in contrast to the boundary wavelets in the above Theorem 5.51. This might be of some interest in applications but it is not the main point of Theorem 5.27. One obtains by the same arguments Corollary 5.30 what cannot be expected by possible modifications of Theorem 5.51 and its proof. Furthermore, Theorem 5.27 paves the way to obtain wavelet bases at least in some of these spaces.

5.4.2 Wavelet frames in (ε, δ)-domains

Let Ω be a bounded (ε, δ)-domain according to Definition 3.1 (i). Then it follows from Proposition 3.6 (i) that Ω is a bounded I-thick domain with $|\partial\Omega| = 0$. Furthermore, Ω is interior regular according to (4.95). But these observations ensure that the proof of Theorem 5.51 can be carried over to the following situation. Let $A^s_{pq}(\Omega)$, $a^s_{pq}(\mathbb{Z}^\Omega)$, the use of the equivalence $\sim$, and the numbers σ_p σ_{pq} be as in the preceding Section 5.4.1.

Theorem 5.53. *Let Ω be a bounded (ε, δ)-domain in $\mathbb{R}^n$ with $n \ge 2$ according to Definition* 3.1 (i). *For any $u \in \mathbb{N}$ there is a u-wavelet system $\{\Phi^j_l\}$ according to Definition* 5.25 (i) *with the following properties.*

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, and $u > s > \sigma_p$. Then $f \in D'(\Omega)$ is an element of $B^s_{pq}(\Omega)$ if, and only if, it can be represented as* (5.249). *One has* (5.250). *Any $f \in B^s_{pq}(\Omega)$ can be represented by* (5.251) *with* (5.252) *(u-wavelet frame).*

(ii) *Let $0 < p < \infty$, $0 < q \le \infty$, and $u > s > \sigma_{pq}$. Then $f \in D'(\Omega)$ is an element of $F^s_{pq}(\Omega)$ if, and only if, it can be represented by* (5.254). *One has* (5.255). *Any $f \in F^s_{pq}(\Omega)$ can be represented by* (5.251) *with* (5.256) *(u-wavelet frame).*

Proof. As remarked above Ω is interior regular according to (4.95). Then it follows from (4.99), (4.101), based on (4.97), (4.98), by the same arguments as in the proof of Theorem 5.51 that (5.249), (5.254) are atomic decompositions. One obtains (5.259). Again by the above references Ω is also a bounded I-thick domain with $|\partial\Omega| = 0$. Then it follows from Theorem 4.4 that there are linear and bounded extension operators (4.16), (4.17). Now one can argue as in the proof of Theorem 5.51. □

Remark 5.54. One can extend part (ii) to $L_p(\Omega) = F^0_{p,2}(\Omega)$ with $1 < p < \infty$. But this is covered in a better way by Theorem 2.36.

Chapter 6
Complements

6.1 Spaces on cellular domains

6.1.1 Riesz bases

It is the main aim of Section 6.1 to prove Theorem 5.49 and, as a consequence, Theorem 5.47. But first we describe the background and rephrase for this purpose some of the previous notation and assertions (also to make Section 6.1 to some extent independently readable).

Definition 6.1. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition 3.4 (iii) or a bounded interval in $\mathbb{R}$ if $n = 1$. Let

$$\mathbb{Z}^\Omega = \big\{ x^j_l \in \Omega : \ j \in \mathbb{N}_0; \ l = 1, \ldots, N_j \big\}, \tag{6.1}$$

typically with $N_j \sim 2^{jn}$, such that for some $c_1 > 0$,

$$|x^j_l - x^j_{l'}| \geq c_1 \, 2^{-j}, \quad j \in \mathbb{N}_0, \ l \neq l'. \tag{6.2}$$

For some $c_2 > 0$ let χ_{jl} be the characteristic function of the ball $B\big(x^j_l, c_2 2^{-j}\big) \subset \mathbb{R}^n$ (centred at x^j_l and of radius $c_2 2^{-j}$). Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. Then $b^s_{pq}(\mathbb{Z}^\Omega)$ is the collection of all sequences

$$\lambda = \big\{ \lambda^j_l \in \mathbb{C} : \ j \in \mathbb{N}_0; \ l = 1, \ldots, N_j \big\} \tag{6.3}$$

such that

$$\| \lambda \, | b^s_{pq}(\mathbb{Z}^\Omega) \| = \bigg(\sum_{j=0}^\infty 2^{j(s - \frac{n}{p})q} \Big(\sum_{l=1}^{N_j} |\lambda^j_l|^p \Big)^{q/p} \bigg)^{1/q} < \infty, \tag{6.4}$$

and $f^s_{pq}(\mathbb{Z}^\Omega)$ is the collection of all sequences (6.3) such that

$$\| \lambda \, | f^s_{pq}(\mathbb{Z}^\Omega) \| = \bigg\| \Big(\sum_{j=0}^\infty \sum_{l=1}^{N_j} 2^{jsq} |\lambda^j_l \chi_{jl}(\cdot)|^q \Big)^{1/q} | L_p(\Omega) \bigg\| < \infty \tag{6.5}$$

(obviously modified if $p = \infty$ and/or $q = \infty$).

Remark 6.2. This coincides essentially with Definition 5.23 (inserted here for sake of completeness). As mentioned in Remark 5.24 with a reference to [T06], Section 1.5.3, pp. 18–19, for any $c'_2 > 0$ one can replace χ_{jl} in (6.5) by the characteristic function of

$$B(x^j_l, c'_2 2^{-j}) \quad \text{or of} \quad B(x^j_l, c'_2 2^{-j}) \cap \Omega$$

(equivalent quasi-norms). If no specification is required we write

$$a^s_{pq}(\mathbb{Z}^\Omega) \quad \text{with } a \in \{b, f\}. \tag{6.6}$$

The spaces $C^u(\Omega)$ have the same meaning as in Definition 5.17.

Definition 6.3. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition 3.4 (iii) or a bounded interval in $\mathbb{R}$ if $n = 1$. Let $\Gamma = \partial\Omega$ and let $\mathbb{Z}^\Omega$ be as in (6.1), (6.2). Let $u \in \mathbb{N}$.

(i) Then

$$\Phi = \left\{\Phi^j_l : j \in \mathbb{N}_0;\ l = 1, \dots, N_j\right\} \subset C^u(\Omega) \tag{6.7}$$

is called a u-wavelet system in $\bar{\Omega}$ if for some $c_3 > 0$ and $c_4 > 0$,

$$\operatorname{supp} \Phi^j_l \subset B\left(x^j_l, c_3 2^{-j}\right) \cap \bar{\Omega}, \quad j \in \mathbb{N}_0;\ l = 1, \dots, N_j, \tag{6.8}$$

and

$$\left|D^\alpha \Phi^j_l(x)\right| \leq c_4\, 2^{j\frac{n}{2} + j|\alpha|}, \quad j \in \mathbb{N}_0;\ l = 1, \dots, N_j,\ x \in \Omega, \tag{6.9}$$

for all $\alpha \in \mathbb{N}^n_0$ with $0 \leq |\alpha| \leq u$.

(ii) The above u-wavelet system is called oscillating if there are positive numbers c_5, c_6, c_7 with $c_6 < c_7$ such that

$$\operatorname{dist}(B(x^0_l, c_3), \Gamma) \geq c_6, \quad l = 1, \dots, \mathbb{N}_0; \tag{6.10}$$

and

$$\left|\int_\Omega \psi(x)\, \Phi^j_l(x)\, \mathrm{d}x\right| \leq c_5\, 2^{-j\frac{n}{2} - ju}\, \|\psi \,|C^u(\Omega)\|, \quad \psi \in C^u(\Omega), \tag{6.11}$$

for all Φ^j_l with $j \in \mathbb{N}$ and

$$\operatorname{dist}(B(x^j_l, c_3 2^{-j}), \Gamma) \notin (c_6 2^{-j}, c_7 2^{-j}). \tag{6.12}$$

(iii) An oscillating u-wavelet system according to part (ii) is called interior if, in addition,

$$\operatorname{dist}(B(x^j_l, c_3 2^{-j}), \Gamma) \geq c_6\, 2^{-j}, \quad j \in \mathbb{N}_0;\ l = 1, \dots, N_j. \tag{6.13}$$

Remark 6.4. Parts (i) and (ii) coincide essentially with Definition 5.25 (again inserted here for sake of completeness). The oscillation (6.11) is not required for the terms with $j = 0$ and also not for the wavelets Φ^j_l with $j \in \mathbb{N}$ and

$$\operatorname{dist}(x^j_l, \Gamma) \sim \operatorname{dist}(\operatorname{supp} \Phi^j_l, \Gamma) \sim 2^{-j}, \tag{6.14}$$

roughly speaking. One may ask for a weaker version restricting (6.11) to all Φ^j_l with

$$\operatorname{dist}(\operatorname{supp} \Phi^j_l, \Gamma) \geq c_6 2^{-j}, \quad j \in \mathbb{N};\ l = 1, \dots, N_j. \tag{6.15}$$

Boundary wavelets with $\Gamma \cap \operatorname{supp} \Phi_l^j \neq \emptyset$ would no longer assumed to oscillate. But (almost) all of our constructions so far and also in the sequel produce the above sharper version. The only exceptions are the u-wavelet systems in Section 5.4. But they are somewhat outside of our main interests. One may consult Remark 5.52. Similarly one may call a u-wavelet system interior if one has

$$\operatorname{supp} \Phi_l^j \subset \Omega, \quad j \in \mathbb{N}_0;\ l = 1, \dots, N_j, \tag{6.16}$$

instead of the stronger assumption (6.13). But all our constructions of u-wavelet bases or u-wavelet frames with (6.16) satisfy automatically (6.13).

We recalled at the beginning of Section 5.2.1 what is meant by a *Riesz basis* in $L_2(\Gamma)$. The related sequence space is ℓ_2. Furthermore we refer to Section 1.2.2 for a definition of (unconditional) bases in complex quasi-Banach spaces. There is the following natural generalisation of Riesz bases. Let $a^s_{pq}(\mathbb{Z}^\Omega)$ be as in (6.6) and Definition 6.1.

Definition 6.5. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition 3.4 (iii) or a bounded interval on $\mathbb{R}$ if $n = 1$. Let $0 < p, q < \infty$ and $s \in \mathbb{R}$. Let $\hat{A}^s_{pq}(\Omega)$ be either $A^s_{pq}(\Omega)$ or $\tilde{A}^s_{pq}(\Omega)$ according to Definition 2.1. An [oscillating]{interior} u-wavelet system $\Phi = \{\Phi_l^j\} \subset \hat{A}^s_{pq}(\Omega)$ as introduced in Definition 6.3 (i)[(ii)]{(iii)} is called an [oscillating]{interior} u-Riesz basis for $\hat{A}^s_{pq}(\Omega)$ if it has the following properties:

1. An element $f \in D'(\Omega)$ belongs to $\hat{A}^s_{pq}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^\infty \sum_{l=1}^{N_j} \lambda_l^j \, 2^{-jn/2} \, \Phi_l^j, \quad \lambda \in a^s_{pq}(\mathbb{Z}^\Omega), \tag{6.17}$$

   unconditional convergence being in $\hat{A}^s_{pq}(\Omega)$.

2. The representation (6.17) is unique,

$$f = \sum_{j=0}^\infty \sum_{l=1}^{N_j} \lambda_l^j(f) \, 2^{-jn/2} \, \Phi_l^j, \tag{6.18}$$

   where $\lambda_l^j(\cdot) \in \hat{A}^s_{pq}(\Omega)'$ are linear and continuous functionals on $\hat{A}^s_{pq}(\Omega)$.

3. Furthermore,

$$f \mapsto \{\lambda_l^j(f)\} \text{ is an isomorphic map of } \hat{A}^s_{pq}(\Omega) \text{ onto } a^s_{pq}(\mathbb{Z}^\Omega). \tag{6.19}$$

Remark 6.6. It is the main aim of this Section 6.1 to have a closer look at wavelet bases in spaces $A^s_{pq}(\Omega)$ where Ω is a cellular domain and $p < \infty$, $q < \infty$. As will be recalled below a cellular domain is a special bounded Lipschitz domain. This may explain our restriction in the above definition to spaces with $p < \infty$, $q < \infty$ in bounded Lipschitz domains. It is quite obvious that one can extend the above definition of u-Riesz bases to spaces $\hat{A}^s_{pq}(\Omega)$ with $p < \infty$, $q < \infty$ in arbitrary domains Ω in $\mathbb{R}^n$, Definition 2.1, or in diverse types of thick domains, Definition 3.1. Furthermore, parallel to [oscillating]{interior} u-Riesz bases one can introduce [oscillating]{interior} *u-Riesz frames*. Then one can also admit $q = \infty$ and, for the B-spaces, $p = \infty$. This can be done in a rather obvious way, where Remark 5.11 suggests to incorporate in a corresponding definition both *stability* and *optimality*. Furthermore there are natural counterparts of *u-Riesz bases* and *u-Riesz frames* for spaces $A^s_{pq}(\Gamma)$ on compact C^∞ manifolds according to Definitions 5.1, 5.5. We stick at the above definition illustrated by the following examples.

Theorem 6.7. (i) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition* 3.4 (iii). *Then any space $A^s_{pq}(\Omega)$, $\tilde{A}^s_{pq}(\Omega)$ covered by Definition* 3.11 *with $p < \infty$, $q < \infty$ has for $u \in \mathbb{N}$ with*

$$u > \max(s, \sigma_p - s) \text{ for } B\text{-spaces}; \quad u > \max(s, \sigma_{pq} - s) \text{ for } F\text{-spaces}, \tag{6.20}$$

an interior u-Riesz basis.

(ii) *Let Ω be a bounded C^∞ domain in the plane $\mathbb{R}^2$ according to Definition* 3.4 (iii). *Then any space $A^s_{pq}(\Omega)$ with* (5.190) *has for any $u \in \mathbb{N}$ with $u > s$ an oscillating u-Riesz basis.*

(iii) *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ with $n \geq 3$ according to Definition* 3.4 (iii) *such that each connected boundary component is diffeomorphic to the sphere $\mathbb{S}^{n-1}$. Then any space $A^s_{pq}(\Omega)$ with* (5.215) *and $0 < q < \infty$ for B-spaces, $1 \leq q < \infty$ for F-spaces has for any $u \in \mathbb{N}$ with $u > s$ an oscillating u-Riesz basis.*

Proof. These are special cases and reformulations of Theorems 3.13, 3.23, 5.35, 5.38. In the related formulation one can replace $r < u$ in Theorem 5.35 by $s < u$. □

Remark 6.8. We rephrased the previous assertions to provide a better understanding of what follows. It is one of the main aims of this Section 6.1 to get rid of the additional assumption in part (iii) of the above theorem that the connected boundary components of $\partial\Omega$ are diffeomorphic to $\mathbb{S}^{n-1}$. In case of cellular domains (as recalled below) we have so far interior u-Riesz bases for the spaces covered by Theorem 5.43. In what follows it is our main goal to prove Theorems 5.47 and 5.49. But first we have a closer look at some peculiarities for spaces of type $A^s_{pq}(\Omega)$ in cellular domains also for its own sake.

6.1.2 Basic properties

In Definition 5.17 we introduced the spaces $C^\infty(\Omega)$ in domains Ω in $\mathbb{R}^n$. Recall that a one-to-one *map* ψ from a bounded domain Ω in $\mathbb{R}^n$ onto a bounded domain ω in $\mathbb{R}^n$,

$$\psi : \Omega \ni x \mapsto y = \psi(x) \in \omega, \tag{6.21}$$

is called a *diffeomorphic map* if

$$\psi_l \in C^\infty(\Omega) \quad \text{and} \quad (\psi^{-1})_l \in C^\infty(\omega), \quad l = 1, \ldots, n, \tag{6.22}$$

for the components ψ_l of ψ and $(\psi^{-1})_l$ of its inverse ψ^{-1},

$$\psi^{-1} \circ \psi = \text{id in } \Omega \quad \text{and} \quad \psi \circ \psi^{-1} = \text{id in } \omega.$$

Let again

$$Q = \{x \in \mathbb{R}^n : 0 < x_l < 1\} \tag{6.23}$$

be the unit cube in $\mathbb{R}^n$. We explained at the beginning of Section 5.3.3 what is meant by a *polyhedron* with Q in (6.23) as a proto-type. A bounded domain Ω in $\mathbb{R}^n$ is said to be *diffeomorphic* to a bounded domain ω in $\mathbb{R}^n$ if there is a diffeomorphic map ψ of a neighbourhood of $\overline{\Omega}$ onto a neighbourhood of $\overline{\omega}$ with $\omega = \psi(\Omega)$. If Γ is a set in $\mathbb{R}^n$ then Γ° is the largest open set in $\mathbb{R}^n$ with $\Gamma^\circ \subset \Gamma$ (the interior of Γ).

Definition 6.9. A domain Ω in $\mathbb{R}^n$, $n \geq 2$, is said to be cellular if it is a bounded Lipschitz domain according to Definition 3.4 (iii) which can be represented as

$$\Omega = \Big(\bigcup_{l=1}^{L} \overline{\Omega_l} \Big)^\circ, \quad \text{with } \Omega_l \cap \Omega_{l'} = \emptyset \text{ if } l \neq l', \tag{6.24}$$

such that each Ω_l is diffeomorphic to a polyhedron.

Remark 6.10. This coincides essentially with Definition 5.40 (i) inserted here for sake of completeness. Since $\psi^l(\Omega_l)$ is a domain it follows that Ω_l is also a domain.

Proposition 6.11. *A bounded C^∞ domain in $\mathbb{R}^n$ according to Definition 3.4 (iii), $n \geq 2$, is cellular.*

Proof. This follows by direct arguments. It has also been mentioned in Remark 5.41. □

In (3.84) we recalled what is meant by a *pointwise multiplier* for $A^s_{pq}(\mathbb{R}^n)$ with a reference to [RuS96] for explanations. We are interested in characteristic functions χ_Ω of domains Ω as pointwise multipliers.

Proposition 6.12. *Let Ω be a cellular domain in $\mathbb{R}^n$ according to Definition 6.9. Let $A^s_{pq}(\mathbb{R}^n)$ and $A^s_{pq}(\Omega)$ be the spaces introduced in Definitions 1.1, 2.1.*

(i) *Then χ_Ω is a pointwise multiplier in $A^s_{pq}(\mathbb{R}^n)$ if, and only if,*

$$0 < p \leq \infty, \quad 0 < q \leq \infty, \quad \max\Big(n\Big(\frac{1}{p} - 1\Big), \frac{1}{p} - 1 \Big) < s < \frac{1}{p}, \tag{6.25}$$

($p < \infty$ for the F-spaces).

(ii) *Let $0 < p \leq \infty$ and $0 < q \leq \infty$ (with $p < \infty$ for the F-spaces). Let $\sigma \in \mathbb{R}$, $k \in \mathbb{N}$ and $s = \sigma + k$. Then*

$$A^s_{pq}(\Omega) = \big\{ f \in A^\sigma_{pq}(\Omega) : D^\alpha f \in A^\sigma_{pq}(\Omega),\ |\alpha| \leq k \big\} \tag{6.26}$$

and

$$\|f \,|A^s_{pq}(\Omega)\| \sim \sum_{|\alpha|\le k} \|D^\alpha f \,|A^\sigma_{pq}(\Omega)\| \tag{6.27}$$

(*equivalent quasi-norms*).

Proof. The assertion (5.222), (5.223) about characteristic functions of half-spaces as pointwise multipliers in $A^s_{pq}(\mathbb{R}^n)$ carries over to cubes, polyhedrons, and cellular domains. There one finds also the necessary references. Part (ii) is a special case of Proposition 4.21. □

Proposition 6.13. *Let Ω be a cellular domain in $\mathbb{R}^n$ according to Definition 6.9.*

(i) *Let p, q, s be as in (6.25) with $p < \infty$ for the F-spaces. Then*

$$A^s_{pq}(\Omega) = \tilde{A}^s_{pq}(\Omega) = \tilde{A}^s_{pq}(\bar{\Omega}) \tag{6.28}$$

for the corresponding spaces from Definition 2.1 and the interpretation given in Remark 2.2.

(ii) *Let*

$$0 < p < \infty, \quad 0 < q < \infty, \quad \max\Big(n\Big(\frac{1}{p}-1\Big), \frac{1}{p}-1\Big) < s < \frac{1}{p}. \tag{6.29}$$

Then

$$A^s_{pq}(\Omega) = \tilde{A}^s_{pq}(\Omega) = \mathring{A}^s_{pq}(\Omega) \tag{6.30}$$

for the corresponding spaces from Definitions 2.1, 5.17 (i).

Proof. Step 1. According to Remark 2.2 the second equality in (6.28) is justified if

$$\big\{g \in A^s_{pq}(\mathbb{R}^n) : \ \operatorname{supp} g \subset \partial\Omega\big\} = \{0\}. \tag{6.31}$$

Recall that

$$A^s_{pq}(\Omega) \hookrightarrow L_1(\Omega) \quad \text{if } s > \sigma_p = n\big(\tfrac{1}{p}-1\big)_+. \tag{6.32}$$

Since $|\partial\Omega| = 0$ one obtains (6.31) in these cases. It remains to prove (6.31) for the admitted spaces with $s \le 0$. It is sufficient to deal with

$$B^s_{pp}(\mathbb{R}^n), \quad 1 < p < \infty, \ \tfrac{1}{p} - 1 < s < 0. \tag{6.33}$$

We use duality,

$$B^s_{pp}(\mathbb{R}^n)' = B^{-s}_{p'p'}(\mathbb{R}^n), \quad \tfrac{1}{p} + \tfrac{1}{p'} = 1, \ 0 < -s < \tfrac{1}{p'}, \tag{6.34}$$

[T83], Section 2.11.2. Let $\varphi \in S(\mathbb{R}^n)$. Then it follows from Proposition 6.12 (i) that both $\varphi\chi_\Omega$ and $\varphi(1-\chi_\Omega)$ belong to $B^{-s}_{p'p'}(\mathbb{R}^n)$. On the other hand, the translation

$$g(\cdot) \mapsto g_h(\cdot) = g(\cdot + h) \quad \text{with } h \in \mathbb{R}^n, \tag{6.35}$$

is continuous in $B^{-s}_{p'p'}(\mathbb{R}^n)$. To justify this claim we recall that $D(\mathbb{R}^n)$ is dense in $B^{-s}_{p'p'}(\mathbb{R}^n)$. If $\psi \in D(\mathbb{R}^n)$ approximates $g \in B^{-s}_{p'p'}(\mathbb{R}^n)$ then ψ_h approximates g_h (uniformly in h) and the claimed continuity of the translation operator follows from

$$\begin{aligned}
&\|f - f_h \,|B^{-s}_{p'p'}(\mathbb{R}^n)\| \\
&\quad \le c\,\|f-\psi\,|B^{-s}_{p'p'}(\mathbb{R}^n)\| + c\,\|\psi-\psi_h\,|B^{-s}_{p'p'}(\mathbb{R}^n)\| + c\,\|(f-\psi)_h\,|B^{-s}_{p'p'}(\mathbb{R}^n)\| \\
&\quad \le c'\,\|f-\psi\,|B^{-s}_{p'p'}(\mathbb{R}^n)\| + c'\|\psi-\psi_h\,|B^{-s}_{p'p'}(\mathbb{R}^n)\|.
\end{aligned} \tag{6.36}$$

Then one has that $D(\mathbb{R}^n \setminus \partial\Omega)$ is dense in $B^{-s}_{p'p'}(\mathbb{R}^n)$. Let now $g \in B^s_{pp}(\mathbb{R}^n)$ with $\operatorname{supp} g \subset \partial\Omega$ and let $\varphi \in S(\mathbb{R}^n)$ be approximated in $B^{-s}_{p'p'}(\mathbb{R}^n)$ by $\varphi^k \in D(\mathbb{R}^n \setminus \partial\Omega)$. Now one obtains that

$$g(\varphi) = \lim_{k\to\infty} g(\varphi^k) = 0 \quad \text{for any } \varphi \in D(\mathbb{R}^n). \tag{6.37}$$

Hence $g = 0$. We obtain (6.31). This justifies the second equality in (6.28). The first equality follows now from Proposition 6.12 (i).

Step 2. If $p < \infty$, $q < \infty$ then one obtains from the standard localisation (multiplication with a smooth resolution of unity) and the continuity of the translation operator as indicated above that $D(\Omega)$ is dense in $\tilde{A}^s_{pq}(\Omega)$. This proves (6.30). □

Remark 6.14. Assertions of type (6.28), (6.30) under the indicated conditions for the parameters p, q, s have some history. As far as bounded C^∞ domains are concerned one may consult [T01], Sections 5.3–5.6, pp. 44–50, where also the above duality argument comes from. In case of bounded Lipschitz domains we have Proposition 5.19 with the restriction $q \ge \min(p,1)$ for the F-spaces based on the references given in Remark 5.20. It is not known whether one has a full counterpart of the pointwise multiplier assertion in Proposition 6.12 (i) and, as a consequence, of Proposition 6.13 for bounded Lipschitz domains. As far as pointwise multipliers are concerned one may consult [FrJ90], Corollary 13.6, and [Tri02]. This may support even at this level (before discussing boundary values of derivatives) to specify bounded Lipschitz domains to cellular domains which are bounded by definition. In Proposition 3.19 we described a counterpart for more general domains.

Proposition 6.15. *Let Ω be a cellular domain in $\mathbb{R}^n$ according to Definition 6.9. Let $k \in \mathbb{N}$ and*

$$0 < p < \infty, \quad 0 < q < \infty, \quad \max\Big(n\Big(\frac{1}{p}-1\Big), \frac{1}{p}-1\Big) < \sigma < \frac{1}{p}. \tag{6.38}$$

Then

$$\tilde{A}^s_{pq}(\Omega) = \mathring{A}^s_{pq}(\Omega) \quad \text{with } s = \sigma + k \tag{6.39}$$

(equivalent quasi-norms) for the corresponding spaces in Definitions 2.1, 5.17.

Proof. If $f \in \tilde{A}^s_{pq}(\Omega)$ then it follows from Proposition 6.12 (ii) and its $\mathbb{R}^n$-counterpart that

$$D^\alpha f \in \tilde{A}^\sigma_{pq}(\Omega) = A^\sigma_{pq}(\Omega), \quad |\alpha| \le k, \tag{6.40}$$

(equivalent quasi-norms). Now one obtains (6.39) from (6.30) and the translation argument in connection with (6.36) applied to $f \in \tilde{A}^s_{pq}(\Omega)$ extended by zero to $\mathbb{R}^n$ and its derivatives. □

Remark 6.16. Assertions of this type for Sobolev spaces and classical Besov spaces as mentioned in Remark 1.2 (with Ω in place of $\mathbb{R}^n$) in bounded C^∞ domains are known since a long time and may be found in [T78], Section 4.3.2. The extension of these assertions to other spaces A^s_{pq}, especially with $p < 1$, is a somewhat tricky game. One may consult [T06], Section 1.11.6, the above Remark 5.20 and the references given there.

6.1.3 A model case: traces and extension

At the end we wish to extend Theorem 6.7 (iii) to cellular domains without any additional assumptions about the boundary. This will be done by the same method as in Chapter 5 based on traces, wavelet-friendly extensions and related decompositions as in Theorems 5.14, 5.21. But now one has to care for faces and edges of several dimensions which complicates the situation. We deal first with a model case.

Let $n \in \mathbb{N}$ and $l \in \mathbb{N}$ with $l < n$. Let $\mathbb{R}^n = \mathbb{R}^l \times \mathbb{R}^{n-l}$ with $x = (y, z) \in \mathbb{R}^n$,

$$y = (y_1, \ldots, y_l) \in \mathbb{R}^l, \quad z = (z_1, \ldots, z_{n-l}) \in \mathbb{R}^{n-l}, \tag{6.41}$$

where $\mathbb{R}^l$ is identified with the hyper-plane $x = (y, 0)$ in $\mathbb{R}^n$. Let Q_l be the unit cube

$$Q_l = \big\{x = (y, z) \in \mathbb{R}^n : \ z = 0, \ 0 < y_m < 1; \ m = 1, \ldots, l\big\} \tag{6.42}$$

in this hyper-plane. Let

$$Q^n_l = \big\{x = (y, z) \in \mathbb{R}^n, \ y \in Q_l, \ z \in \mathbb{R}^{n-l}\big\} \tag{6.43}$$

be the related cylindrical domain in $\mathbb{R}^n$. Let tr_l be the trace operator

$$\mathrm{tr}_l : \ f(x) \mapsto f(y, 0), \quad f \in A^s_{pq}(\mathbb{R}^n), \tag{6.44}$$

on $\mathbb{R}^l$ or Q_l (if exists) defined by obvious modification of (5.5) and the related comments in Section 5.1.1. Let

$$\mathbb{N}^n_l = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_0 : \ \alpha_1 = \cdots = \alpha_l = 0\}. \tag{6.45}$$

Then

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_{l+1}} \ldots \partial z_n^{\alpha_n}}, \quad \alpha \in \mathbb{N}^n_l, \tag{6.46}$$

are the derivatives perpendicular to Q_l. Let

$$\sigma^l_p = l\Big(\frac{1}{p} - 1\Big)_+, \quad 0 < p \le \infty, \ l \in \mathbb{N}, \tag{6.47}$$

be as in (4.15), now indicating l. Let $A^s_{pq}(\mathbb{R}^n)$ be the spaces as introduced in (1.95), Definition 1.1, and $B^\sigma_{pq}(Q_l)$ be the spaces from Definition 2.1 in the context of $\mathbb{R}^l$. We need an extension of the trace assertion (5.59)–(5.63) replacing the $(n-1)$-dimensional manifold Γ by Q_l. The classical case, covering the Sobolev spaces $H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$, $1 < p < \infty$, and the Besov spaces $B^s_{pq}(\mathbb{R}^n)$, $1 < p < \infty$, $1 \le q \le \infty$ may be found in [T78], Section 2.9.4, 4.7.2, where we described also the rich history of this problem. The full assertion for all relevant spaces $B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$ can be obtained by iteration of [T83], Theorem 3.3.3, p. 200, or nowadays by atomic or wavelet arguments. For our later purpose $p \ge 1$ would be sufficient. But as in Section 5.1.3 traces and wavelet-friendly extensions are of interest for its own sake and a complete description is desirable.

Proposition 6.17. *Let $l \in \mathbb{N}$, $n \in \mathbb{N}$ with $l < n$, and $r \in \mathbb{N}_0$. Let $0 < p \le \infty$ ($p < \infty$ for the F-spaces), $0 < q \le \infty$, and*

$$s > r + \frac{n-l}{p} + \sigma^l_p. \tag{6.48}$$

Let Q_l be as in (6.42). *Let* tr^r_l,

$$\mathrm{tr}^r_l : f \mapsto \big\{ \mathrm{tr}_l\, D^\alpha f : \alpha \in \mathbb{N}^n_l,\ |\alpha| \le r \big\}. \tag{6.49}$$

Then

$$\mathrm{tr}^r_l : B^s_{pq}(\mathbb{R}^n) \hookrightarrow \prod_{\substack{\alpha\in\mathbb{N}^n_l \\ |\alpha|\le r}} B^{s-\frac{n-l}{p}-|\alpha|}_{pq}(Q_l) \tag{6.50}$$

and

$$\mathrm{tr}^r_l : F^s_{pq}(\mathbb{R}^n) \hookrightarrow \prod_{\substack{\alpha\in\mathbb{N}^n_l \\ |\alpha|\le r}} B^{s-\frac{n-l}{p}-|\alpha|}_{pp}(Q_l) \tag{6.51}$$

(*continuous embedding*).

Remark 6.18. We refer to the above comments. But one can prove these embeddings directly restricting atomic expansions of $f \in A^s_{pq}(\mathbb{R}^n)$ to $\mathbb{R}^l$. No moment conditions are needed in the counterparts of (5.64)–(5.67).

Crucial for what follows is the modification of the wavelet-friendly extension operator as constructed in (5.75) and used in Theorem 5.14. Let $C^u(Q_l)$ with $u \in \mathbb{N}$ be as in Definition 5.17 (ii) and let

$$\big\{ \Phi^j_m(y) : \ j \in \mathbb{N}_0;\ m = 1, \ldots, N_j \big\} \subset C^u(Q_l) \tag{6.52}$$

with $N_j \sim 2^{jl}$ be an interior orthonormal u-wavelet basis in $L_2(Q_l)$ according to Theorem 2.33 and Definitions 2.31, 2.4 (with $\mathbb{R}^l$ in place of $\mathbb{R}^n$). We apply Theorem 3.13 to

$$\widetilde{B}^\sigma_{pq}(Q_l), \quad 0 < p \le \infty, \quad 0 < q \le \infty, \quad \sigma^l_p < \sigma < u. \tag{6.53}$$

Then one has for $g \in \widetilde{B}^{\sigma}_{pq}(Q_l)$ the wavelet expansion

$$g = \sum_{j=0}^{\infty} \sum_{m=1}^{N_j} \lambda^j_m(g)\, 2^{-jl/2}\, \Phi^j_m \tag{6.54}$$

with

$$\lambda^j_m(g) = 2^{jl/2} \int_{Q_l} g(y)\, \Phi^j_m(y)\, \mathrm{d}y \tag{6.55}$$

and the isomorphic map

$$g \mapsto \{\lambda^j_m(g)\} \quad \text{of } \widetilde{B}^{\sigma}_{pq}(Q_l) \text{ onto } b^{\sigma}_{pq}(\mathbb{Z}^{Q_l}). \tag{6.56}$$

Here $b^{\sigma}_{pq}(\mathbb{Z}^{Q_l})$ comes from (6.4). If $p < \infty, q < \infty$ then $\{\Phi^j_m\}$ in (6.52) is an interior u-Riesz basis in $\widetilde{B}^{\sigma}_{pq}(Q_l)$ according to Definition 6.5. We refer to Theorem 3.13 where we clarified the convergence of (6.54) also in the cases where $p = \infty$ and/or $q = \infty$. Next we modify the construction (5.75) of an extension operator such that it can be applied to the right-hand sides of (6.50), (6.51). Let

$$\chi \in D(\mathbb{R}^{n-l}), \quad \operatorname{supp} \chi \subset \{z \in \mathbb{R}^{n-l} : |z| \le 2\}, \quad \chi(z) = 1 \text{ if } |z| \le 1, \tag{6.57}$$

with $z \in \mathbb{R}^{n-l}$ as in (6.41). One may assume that χ has sufficiently many moment conditions,

$$\int_{\mathbb{R}^{n-l}} \chi(z)\, z^{\beta}\, \mathrm{d}z = 0 \quad \text{if } |\beta| \le L. \tag{6.58}$$

Let Φ^j_m be the wavelets according to (6.52) and

$$\Phi^{j,\alpha}_m(x) = 2^{j|\alpha|} z^{\alpha}\, \chi(2^j z)\, 2^{(n-l)j/2}\, \Phi^j_m(y), \quad \alpha \in \mathbb{N}^n_l, \tag{6.59}$$

according to (6.41) and (6.45) be the counterpart of (5.76). Then

$$\begin{aligned} g &= \operatorname{Ext}^{r,u}_l \{g_\alpha : \alpha \in \mathbb{N}^n_l,\ |\alpha| \le r\}(x) \\ &= \sum_{|\alpha| \le r} \sum_{j=0}^{\infty} \sum_{m=1}^{N_j} \frac{1}{\alpha!}\, \lambda^j_m(g_\alpha)\, 2^{-j|\alpha|}\, 2^{-jn/2}\, \Phi^{j,\alpha}_m(x) \end{aligned} \tag{6.60}$$

is the counterpart of (5.75) where $\{g_\alpha\} \subset L_1(Q_l)$ and $\lambda^j_m(g_\alpha)$ as in (6.55) (with g_α in place of g). Let $\tilde{A}^s_{pq}(Q^n_l)$ be the spaces as introduced in Definition 2.1 where Q^n_l is the cylindrical domain (6.43).

Theorem 6.19. *Let $\{\Phi^j_m\}$ with $u \in \mathbb{N}$ be the interior orthonormal u-wavelet basis in $L_2(Q_l)$ according to (6.52). Let χ be as in (6.57), (6.58) (with $L \in \mathbb{N}_0$ sufficiently large in dependence on q for the F-spaces). Let $r \in \mathbb{N}_0$ and let $\operatorname{Ext}^{r,u}_l$ be given by (6.60) with (6.55). Then*

$$\operatorname{Ext}^{r,u}_l : \quad \{g_\alpha : \alpha \in \mathbb{N}^n_l,\ |\alpha| \le r\} \mapsto g \tag{6.61}$$

is an extension operator

$$\mathrm{Ext}_l^{r,u}: \prod_{\substack{\alpha\in\mathbb{N}_l^n \\ |\alpha|\le r}} \widetilde{B}_{pq}^{s-\frac{n-l}{p}-|\alpha|}(Q_l) \hookrightarrow \widetilde{B}_{pq}^{s}(Q_l^n), \qquad 0<p\le\infty, \quad 0<q\le\infty, \quad u>s>r+\tfrac{n-l}{p}+\sigma_p^l, \tag{6.62}$$

and

$$\mathrm{Ext}_l^{r,u}: \prod_{\substack{\alpha\in\mathbb{N}_l^n \\ |\alpha|\le r}} \widetilde{B}_{pp}^{s-\frac{n-l}{p}-|\alpha|}(Q_l) \hookrightarrow \widetilde{F}_{pq}^{s}(Q_l^n), \qquad 0<p<\infty, \quad 0<q\le\infty, \quad u>s>r+\tfrac{n-l}{p}+\sigma_p^l, \tag{6.63}$$

with

$$\mathrm{tr}_l^r \circ \mathrm{Ext}_l^{r,u} = \mathrm{id}, \quad \textit{identity in} \prod_{\substack{\alpha\in\mathbb{N}_l^n \\ |\alpha|\le r}} \widetilde{B}_{pq}^{s-\frac{n-l}{p}-|\alpha|}(Q_l). \tag{6.64}$$

Proof. Since $s>\sigma_p^l$ and, for $p<1$,

$$s > \frac{n-l}{p} + \sigma_p^l = \frac{n}{p} - l > \sigma_p^n, \tag{6.65}$$

it follows from Definition 2.1 and Remark 2.2 that one can identify $\widetilde{B}_{pq}^{\sigma}(Q_l)$ in (6.62), (6.63) with $\widetilde{B}_{pq}^{\sigma}(\overline{Q_l})$, considered as a subspace of $B_{pq}^{\sigma}(\mathbb{R}^l)$, and $\widetilde{A}_{pq}^{s}(Q_l^n)$ with $\widetilde{A}_{pq}^{s}(\overline{Q_l^n})$ as a subspace of $A_{pq}^{s}(\mathbb{R}^n)$. Then the case $l=n-1$ is essentially covered by Theorem 5.14 and its proof. There we dealt with a bounded C^∞ domain Ω in place of Q_l^n and its boundary $\Gamma=\partial\Omega$ in place of Q_{n-1}. Since we replaced $B_{pq}^{\sigma}(\Gamma)$ now by $\widetilde{B}_{pq}^{\sigma}(Q_{n-1})$ we avoid any difficulty which may be caused by the boundary ∂Q_{n-1} of Q_{n-1}. But otherwise one can follow the arguments in the proof of Theorem 5.14 for all $l\in\mathbb{N}$ with $l<n$. In particular the functions $\Phi_m^{j,\alpha}$ in (6.59) are the direct counterpart of (5.76) (not normalised atoms). By (6.65) no moment conditions are needed in case of the B-spaces. If $\varepsilon\le q<\infty$ for some ε with $0<\varepsilon<1$ then one may choose $L\in\mathbb{N}$ with $L\ge n\big(\frac{1}{\varepsilon}-1\big)$ in (6.58) in case of the F-spaces. □

Remark 6.20. It comes out that $\mathrm{Ext}_l^{r,u}$ is a common extension operator for given r,u and $0<\varepsilon\le 1$ with $\varepsilon\le q$ in (6.63) in place of $0<q$.

6.1.4 A model case: approximation, density, decomposition

Let Q_l and Q_l^n be as in (6.42), (6.43). By Definition 5.17 (i) the completion of $D(Q_l^n)$ in $A_{pq}^s(Q_l^n)$ is denoted by $\mathring{A}_{pq}^s(Q_l^n)$. One can replace Ω in Proposition 6.15 by Q_l^n.

Then one obtains that

$$\tilde{A}^s_{pq}(Q^n_l) = \mathring{A}^s_{pq}(Q^n_l) \tag{6.66}$$

(equivalent quasi-norms) if

$$1 \le p < \infty, \quad 0 < q < \infty, \quad 0 < s - \frac{1}{p} \not\in \mathbb{N}. \tag{6.67}$$

If $p < 1$ then one has to exclude some (p, s)-regions in order to obtain (6.66) (recall that $n \ge 2$). This is a first hint that it might be reasonable to restrict the further considerations to $p \ge 1$. But we add a second more substantial argument. For $0 < p < \infty$, $0 < q < \infty$ and the largest admitted $r \in \mathbb{N}_0$ in (6.48) one may ask under which circumstances

$$D\big(Q^n_l \setminus Q_l\big) \quad \text{is dense in} \quad \big\{f \in \tilde{A}^s_{pq}(Q^n_l) : \operatorname{tr}^r_l f = 0\big\}. \tag{6.68}$$

Of course, $Q^n_l \setminus Q_l$ is considered as a set in $\mathbb{R}^n$. As for the classical cases we refer to [T78], Section 2.9.4, p. 223, complemented by [T83], Section 3.4.3, p. 210. We begin with a preparation using the same notation as in (6.41)–(6.43) and in Proposition 6.17.

Proposition 6.21. *Let $l \in \mathbb{N}$, $n \in \mathbb{N}$ with $l < n$ and $r \in \mathbb{N}_0$. Let*

$$0 < p < \infty, \quad 0 < q < \infty, \quad s > r + \frac{n-l}{p} + \sigma^l_p. \tag{6.69}$$

Let $u \in \mathbb{N}$ with $u > s$. Then

$$\big\{g \in C^u(\mathbb{R}^n) : \operatorname{supp} g \subset \{y \in Q_l\} \times \{|z| < 1\}, \ \operatorname{tr}^r_l g = 0\big\} \tag{6.70}$$

is dense in

$$\big\{f \in \tilde{A}^s_{pq}(Q^n_l) : \operatorname{tr}^r_l f = 0\big\}. \tag{6.71}$$

Proof. By the same decomposition and translation arguments as in connection with (6.36) now with $h = (h_l, 0)$ parallel to $\mathbb{R}^l$ it follows that it is sufficient to approximate

$$f \in A^s_{pq}(\mathbb{R}^n), \quad \operatorname{supp} f \subset Q^n_l, \quad \operatorname{tr}^r_l f = 0, \tag{6.72}$$

(hence vanishing near ∂Q^n_l). For given $\varepsilon > 0$ one can approximate f in $A^s_{pq}(\mathbb{R}^n)$ by $f_\varepsilon \in S(\mathbb{R}^n)$ with

$$\operatorname{supp} f_\varepsilon \subset Q^n_l \quad \text{and} \quad \|\operatorname{tr}^r_l f_\varepsilon \,|\, \operatorname{tr}^r_l A^s_{pq}(\mathbb{R}^n)\| \le \varepsilon, \tag{6.73}$$

where $\operatorname{tr}^r_l A^s_{pq}(\mathbb{R}^n)$ stands for the trace spaces according to (6.50), (6.51). Let

$$f^\varepsilon = \operatorname{Ext}^{r,u+1}_l \big\{\operatorname{tr}^r_l f_\varepsilon\big\} \tag{6.74}$$

with $\operatorname{Ext}^{r,u+1}_l$ as in (6.60) and Theorem 6.19. By the properties of $\operatorname{Ext}^{r,u+1}_l$ as mentioned in Remark 6.20 one obtains that

$$f^\varepsilon \in C^u(\mathbb{R}^n), \quad \operatorname{supp} f^\varepsilon \subset Q^n_l, \quad \|f^\varepsilon \,|A^s_{pq}(\mathbb{R}^n)\| \le c\varepsilon, \tag{6.75}$$

for some $c > 0$ which is independent of ε. Then $f_\varepsilon - f^\varepsilon$ is the desired approximation. □

Remark 6.22. We return to the question (6.68). By Proposition 6.21 it is sufficient to approximate the function g from (6.70). Let

$$S_j = \{x = (y,z) : \; y \in Q_l, \; 2^{-j-1} \le |z| \le 2^{-j+1}\}, \qquad j \in \mathbb{N}_0, \tag{6.76}$$

and let

$$1 = \sum_{j=0}^{\infty} \sum_{m=1}^{N_j} \varphi_{jm}(x), \quad N_j \sim 2^{jl}, \; x = (y,z), \; y \in Q_l, \; 0 < |z| < 1, \tag{6.77}$$

be a resolution of unity by suitable C^∞ functions φ_{jm} with

$$\operatorname{supp} \varphi_{jm} \subset B_{jm} \subset S_j, \quad j \in \mathbb{N}_0, \; 1 \le m \le N_j, \tag{6.78}$$

where B_{jm} is a ball of radius 2^{-j}. Let g be as in (6.70) (in particular g vanishes near the faces ∂Q_l of Q_l as an $\mathbb{R}^l$-set). Then

$$g = \sum_{j=0}^{\infty} \sum_{m=1}^{N_j} \lambda_{jm} \, 2^{j(r+1)} \, 2^{-j(s-\frac{n}{p})} \, \varphi_{jm} \, g = \sum_{j,m} \lambda_{jm} \, a_{jm} \tag{6.79}$$

is an atomic decomposition of g in $B^s_{pq}(\mathbb{R}^n)$ according to Theorem 1.7 with p, q, s as in (6.69) (not to speak about immaterial constants). By (6.65) no moment conditions are needed. But we used $\operatorname{tr}^r_l g = 0$ according to (6.49) for $g \in C^u(\mathbb{R}^n)$ and related Taylor expansions in z-directions, where the factor $2^{j(r+1)}$ comes from. We have

$$\lambda_{jm} \sim 2^{-j(r+1-s+\frac{n}{p})}. \tag{6.80}$$

Let g_J with $J \in \mathbb{N}$ be given by (6.79) with $j \ge J$ in place of $j \ge 0$. Then it follows by Theorem 1.7 that

$$\begin{aligned} \|g_J \,|B^s_{pq}(\mathbb{R}^n)\| &\le c \Big(\sum_{j=J}^{\infty} \Big(\sum_{m=1}^{N_j} |\lambda_{jm}|^p \Big)^{q/p} \Big)^{1/q} \\ &\le c' \Big(\sum_{j=J}^{\infty} 2^{-jp(r+1-s+\frac{n}{p}-\frac{l}{p})q/p} \Big)^{1/q} \\ &\le c'' \, 2^{-J\delta} \quad \text{if } \delta = r+1-s+\tfrac{n-l}{p} > 0. \end{aligned} \tag{6.81}$$

Then $g - g_J$ approximates g in $B^s_{pq}(\mathbb{R}^n)$. These functions vanish near Q_l. An additional mollification gives (6.68) for $\widetilde{B}^s_{pq}(Q^n_l)$. The corresponding assertion for $\widetilde{F}^s_{pq}(Q^n_l)$ follows from

$$\widetilde{B}^s_{p,\min(p,q)}(Q^n_l) \hookrightarrow \widetilde{F}^s_{pq}(Q^n_l) \tag{6.82}$$

and Proposition 6.21. The largest admitted $r \in \mathbb{N}_0$ in (6.69) is given by

$$r = s - \frac{n-l}{p} - \sigma^l_p - \varepsilon \quad \text{for some } \varepsilon \text{ with } 0 < \varepsilon \le 1. \tag{6.83}$$

If $p \geq 1$ then

$$\delta = 1 - \varepsilon > 0 \quad \text{if, and only if,} \quad s - \frac{n-l}{p} \notin \mathbb{N}. \tag{6.84}$$

If $p < 1$ then (6.81) requires

$$\delta = 1 - \varepsilon - \sigma_p^l = 1 - \varepsilon - l\Big(\frac{1}{p} - 1\Big) > 0. \tag{6.85}$$

Although there are some $p < 1$, $0 < \varepsilon < 1$ with (6.85) the situation is not very satisfactory. Together with (6.66), (6.67) and the comments afterwards it is a second good reason to restrict what follows to $p \geq 1$.

After these discussions we are in a similar situation as in Corollary 5.16 and Theorem 5.21, now with Q_l^n in place of the bounded C^∞ domain Ω and Q_l in place of $\Gamma = \partial\Omega$. We describe briefly the (more or less technical) changes. First we remark that one has by Proposition 6.17 for the same parameters as there the traces

$$\operatorname{tr}_l^r : \widetilde{B}^s_{pq}(Q_l^n) \hookrightarrow \prod_{\substack{\alpha \in \mathbb{N}_l^n \\ |\alpha| \leq r}} \widetilde{B}_{pq}^{s - \frac{n-l}{p} - |\alpha|}(Q_l) \tag{6.86}$$

and

$$\operatorname{tr}_l^r : \widetilde{F}^s_{pq}(Q_l^n) \hookrightarrow \prod_{\substack{\alpha \in \mathbb{N}_l^n \\ |\alpha| \leq r}} \widetilde{B}_{pp}^{s - \frac{n-l}{p} - |\alpha|}(Q_l). \tag{6.87}$$

In the same way as in (5.99), (5.100) one obtains now from Theorem 6.19 for the same parameters p, q, s (and r, u) as there that

$$P_r^l = \operatorname{Ext}_l^{r,u} \circ \operatorname{tr}_l^r : \tilde{A}^s_{pq}(Q_l^n) \hookrightarrow \tilde{A}^s_{pq}(Q_l^n) \tag{6.88}$$

is a projection. Let $P_r^l \tilde{A}^s_{pq}(Q_l^n)$ be its range. Then $T_r^l = \operatorname{id} - P_r^l$ is also a projection and

$$T_r^l \tilde{A}^s_{pq}(Q_l^n) = \big\{ f \in \tilde{A}^s_{pq}(Q_l^n) : \operatorname{tr}_l^r f = 0 \big\} \tag{6.89}$$

is the counterpart of (5.113). As in (5.101) one has the isomorphic maps

$$\left.\begin{aligned} \operatorname{Ext}_l^{r,u} \prod_{\alpha \in \mathbb{N}_l^n, |\alpha| \leq r} \widetilde{B}_{pq}^{s - \frac{n-l}{p} - |\alpha|}(Q_l) &= P_r^l \widetilde{B}^s_{pq}(Q_l^n), \\ \operatorname{Ext}_l^{r,u} \prod_{\alpha \in \mathbb{N}_l^n, |\alpha| \leq r} \widetilde{B}_{pp}^{s - \frac{n-l}{p} - |\alpha|}(Q_l) &= P_r^l \widetilde{F}^s_{pq}(Q_l^n). \end{aligned}\right\} \tag{6.90}$$

Now one can clip together the above considerations. Recall that Q_l and related spaces refer to $\mathbb{R}^l$, whereas Q_l^n and also $Q_l^n \setminus \overline{Q_l}$ and related spaces are considered in $\mathbb{R}^n$. Then one has the following counterpart of Theorem 5.21.

Theorem 6.23. *Let $l \in \mathbb{N}$ and $n \in \mathbb{N}$ with $l < n$. Let*

$$1 \le p < \infty, \quad 0 < q < \infty, \quad 0 < s - \frac{n-l}{p} \notin \mathbb{N}, \quad s - \frac{1}{p} \notin \mathbb{N}. \tag{6.91}$$

Let Q_l and Q_l^n be as in (6.42), (6.43). Then

$$\tilde{A}^s_{pq}(Q^n_l) = \mathring{A}^s_{pq}(Q^n_l). \tag{6.92}$$

Let $r = \left[s - \frac{n-l}{p}\right] \in \mathbb{N}_0$ be the largest integer smaller than $s - \frac{n-l}{p}$. For $s < u \in \mathbb{N}$ let $\mathrm{Ext}^{r,u}_l$ be the extension operator according to (6.60), (6.55) and Theorem 6.19. Then

$$\mathring{B}^s_{pq}(Q^n_l) = \mathring{B}^s_{pq}(Q^n_l \setminus \overline{Q_l}) \times \mathrm{Ext}^{r,u}_l \prod_{\substack{\alpha \in \mathbb{N}^n_l, \\ |\alpha| \le r}} \widetilde{B}_{pq}^{s - \frac{n-l}{p} - |\alpha|}(Q_l) \tag{6.93}$$

and

$$\mathring{F}^s_{pq}(Q^n_l) = \mathring{F}^s_{pq}(Q^n_l \setminus \overline{Q_l}) \times \mathrm{Ext}^{r,u}_l \prod_{\substack{\alpha \in \mathbb{N}^n_l, \\ |\alpha| \le r}} \widetilde{B}_{pp}^{s - \frac{n-l}{p} - |\alpha|}(Q_l) \tag{6.94}$$

(*complemented subspaces*).

Proof. First we remark that (6.92) follows from (6.66), (6.67). Then $\mathring{A}^s_{pq}(\Omega)$ can be decomposed by (6.90) and (6.89) into the second factors on the right-hand sides of (6.93), (6.94) and the space on the right-hand side of (6.89). By Remark 6.22 we have (6.68). This completes the proof of the theorem. □

Corollary 6.24. *Theorem 5.21 remains valid if one replaces (5.115) by*

$$1 \le p < \infty, \quad 0 < q < \infty, \quad -1 < s - \frac{1}{p} \notin \mathbb{N}_0, \tag{6.95}$$

both for B-spaces and F-spaces.

Proof. This follows from Proposition 6.13 (ii) (instead of Proposition 5.19) and the proof of the above theorem with $l = n - 1$. □

Remark 6.25. The restriction $q \ge 1$ for the F-spaces in Theorem 5.21 comes from the reference to [T83] in its proof which can now be improved by the above arguments.

6.1.5 Cubes and polyhedrons: traces and extensions

Let

$$Q = \{x \in \mathbb{R}^n : x = (x_1, \ldots, x_n),\ 0 < x_m < 1;\ m = 1, \ldots, n\} \tag{6.96}$$

be the unit cube in $\mathbb{R}^n$ where $2 \le n \in \mathbb{N}$. We wish to extend Theorem 6.23 from Q_l^n to Q. This is mainly a technical matter, but it requires some care. It is based on the crucial observation that corresponding extension operators of type (6.59), (6.60) from l-dimensional faces of Q of type Q_l into Q do not interfere with the boundary data at all other faces (edges, vertices) of Q. This can be extended to *bounded polyhedral domains* in $\mathbb{R}^n$ without essential changes in a more or less obvious way. One may consider Q as a prototype of a polyhedron in $\mathbb{R}^n$. The boundary $\Gamma = \partial Q$ of Q can be represented as

$$\Gamma = \bigcup_{l=0}^{n-1} \Gamma_l, \quad \Gamma_l \cap \Gamma_{l'} = \emptyset \text{ if } l \ne l', \tag{6.97}$$

where Γ_l collects all l-dimensional faces (edges, vertices). Then Γ_l consists of finitely many l-dimensional cubes of type Q_l as in (6.42) (open as a subset of $\mathbb{R}^l$). We wish to apply Theorem 6.23 where we have now faces of dimensions $l = 0, \ldots, n-1$. This suggests the restriction

$$1 \le p < \infty, \quad 0 < q < \infty, \quad s > \frac{1}{p}, \quad s - \frac{k}{p} \notin \mathbb{N}_0 \text{ for } k = 1, \ldots, n. \tag{6.98}$$

The trace (6.44) is now given by the restriction

$$\mathrm{tr}_l : f(x) \mapsto f|\Gamma_l, \quad f \in A^s_{pq}(\mathbb{R}^n), \tag{6.99}$$

of f to Γ_l. If $\gamma \in \Gamma_l$ the $D^\alpha_\gamma f$ denotes the obvious counterpart of (6.46) where only derivatives are admitted referring to directions perpendicular in $\mathbb{R}^n$ to Γ_l in γ. One replaces (6.49) by

$$\mathrm{tr}^r_l : f \mapsto \big\{ \mathrm{tr}_l \, D^\alpha_\gamma f : |\alpha| \le r \big\}, \quad l = 0, \ldots, n-1, \tag{6.100}$$

for traces on Γ_l. Let $1 \le p < \infty$ and let be either

$$0 < s - \frac{n}{p} \notin \mathbb{N} \quad \text{or} \quad 0 < s - \frac{n - l_0}{p} < \frac{1}{p} \quad \text{for some } l_0 = 1, \ldots, n-1. \tag{6.101}$$

If $s > n/p$ then we put $l_0 = 0$. Recall that $[a]$ is the largest integer smaller than or equal to $a \in \mathbb{R}$. Let

$$\bar{r} = \big(r^{l_0}, \ldots, r^{n-1} \big) \quad \text{with } r^l = \big[s - \tfrac{n-l}{p} \big], \quad l_0 \le l \le n-1. \tag{6.102}$$

Then

$$\mathrm{tr}^{\bar{r}}_\Gamma : f \mapsto \big\{ \mathrm{tr}_l \; D^\alpha_\gamma f : \gamma \in \Gamma_l, \, |\alpha| \le r^l; \, l_0 \le l \le n-1 \big\} \tag{6.103}$$

is the appropriate modification of (6.49).

Proposition 6.26. *Let Q be the cube* (6.96) *with the boundary $\Gamma = \partial Q$ according to* (6.97). *Let*

$$1 \le p < \infty, \quad 0 < q < \infty, \quad s > \frac{1}{p}, \quad s - \frac{k}{p} \notin \mathbb{N}_0 \ \textit{for } k = 1, \ldots, n. \tag{6.104}$$

Let $l_0 = 0$ *if* $s > n/p$ *and otherwise* $l_0 = 1, \ldots, n-1$*, such that*

$$0 < s - \frac{n - l_0}{p} < \frac{1}{p}. \tag{6.105}$$

Let $\mathrm{tr}_p^{\bar r}$ *be as in* (6.102)*,* (6.103)*. Then*

$$\mathrm{tr}_\Gamma^{\bar r} : B^s_{pq}(\mathbb{R}^n) \hookrightarrow \prod_{l=l_0}^{n-1} \prod_{|\alpha| \le r^l} B_{pq}^{s-\frac{n-l}{p}-|\alpha|}(\Gamma_l) \tag{6.106}$$

and

$$\mathrm{tr}_\Gamma^{\bar r} : F^s_{pq}(\mathbb{R}^n) \hookrightarrow \prod_{l=l_0}^{n-1} \prod_{|\alpha| \le r^l} B_{pp}^{s-\frac{n-l}{p}-|\alpha|}(\Gamma_l). \tag{6.107}$$

One can replace $B^s_{pq}(\mathbb{R}^n)$ *by* $B^s_{pq}(Q)$ *and* $F^s_{pq}(\mathbb{R}^n)$ *by* $F^s_{pq}(Q)$*.*

Proof. This follows immediately from Proposition 6.17 and the above considerations. □

Remark 6.27. One can extend the above assertion to $q = \infty$ in (6.104), $p = \infty$ for the B-spaces, and also to $p < 1$ with appropriately adapted restrictions for s as in Proposition 6.17. But our main concern is the counterpart of Theorem 6.23 for the above cube Q. For this purpose one has to clip together the extension operators $\mathrm{Ext}_l^{r,u}$ as used in Theorems 6.19 and 6.23 with a reference to (6.60). Recall that Γ_l consists of l-dimensional (open as subsets of $\mathbb{R}^l$) disjoint cubes of type Q_l in (6.42),

$$\Gamma_l = \overline{\Gamma_l} \setminus \bigcup_{k<l} \Gamma_k, \quad l = 1, \ldots, n-1, \tag{6.108}$$

(corner points if $l = 0$). In particular, $D(\Gamma_l)$ is the disjoint union of sets of type $D(Q_l)$. In the same way one has to interprete $\mathring{B}^\sigma_{pq}(\Gamma_l)$ and $\widetilde{B}^\sigma_{pq}(\Gamma_l)$. With p, q, s as in (6.104) one obtains by Proposition 6.15 that

$$\mathring{B}_{pq}^{s-\frac{n-l}{p}-|\alpha|}(\Gamma_l) = \widetilde{B}_{pq}^{s-\frac{n-l}{p}-|\alpha|}(\Gamma_l), \quad l = 1, \ldots, n-1, \quad |\alpha| \le r^l. \tag{6.109}$$

Let $\mathrm{Ext}_{\Gamma_l}^{r,u}$ be the obvious generalisation of (6.60) from Q_l to Γ_l and let

$$\mathrm{Ext}_\Gamma^{\bar r,u} = \left\{ \mathrm{Ext}_{\Gamma_l}^{r^l,u} : l_0 \le l \le n-1 \right\} \tag{6.110}$$

with $\bar r$ and l_0 as in (6.102). If $s > n/p$ then $l = l_0 = 0$ is admitted. These are the corner points of Q. Then one has to modify (6.59), (6.60) as in Section 5.2.3 where we dealt with the one-dimensional case and traces in points.

Theorem 6.28. *Let Q be the cube in* (6.96) *with the boundary $\Gamma = \partial Q$ according to* (6.97)*. Let*

$$1 \le p < \infty, \quad 0 < q < \infty, \quad s > \frac{1}{p}, \quad s - \frac{k}{p} \notin \mathbb{N}_0 \quad \text{for } k = 1, \ldots, n. \tag{6.111}$$

Let $s < u \in \mathbb{N}$. Let l_0 and $\bar{r}$ be as in (6.101), (6.102) *and Proposition* 6.26*. Let $\mathrm{Ext}^{\bar{r},u}_\Gamma$ be given by* (6.110)*. Then*

$$\mathrm{Ext}^{\bar{r},u}_\Gamma : \big\{ g^l_\alpha : \; l_0 \le l \le n-1, \; |\alpha| \le r^l \big\} \mapsto g \tag{6.112}$$

is an extension operator,

$$\mathrm{Ext}^{\bar{r},u}_\Gamma : \prod_{l=l_0}^{n-1} \prod_{|\alpha| \le r^l} \widetilde{B}^{s-\frac{n-l}{p}-|\alpha|}_{pq}(\Gamma_l) \hookrightarrow B^s_{pq}(Q) \tag{6.113}$$

and

$$\mathrm{Ext}^{\bar{r},u}_\Gamma : \prod_{l=l_0}^{n-1} \prod_{|\alpha| \le r^l} \widetilde{B}^{s-\frac{n-l}{p}-|\alpha|}_{pp}(\Gamma_l) \hookrightarrow F^s_{pq}(Q). \tag{6.114}$$

Let $\mathrm{tr}^{\bar{r}}_\Gamma$ be the trace operator according to (6.103)*. Then*

$$\mathrm{tr}^{\bar{r}}_\Gamma \circ \mathrm{Ext}^{\bar{r},u}_\Gamma = \mathrm{id}, \quad \text{identity in } \prod_{l=l_0}^{n-1} \prod_{|\alpha| \le r^l} \widetilde{B}^{s-\frac{n-l}{p}-|\alpha|}_{pq}(\Gamma_l). \tag{6.115}$$

Furthermore,

$$B^s_{pq}(Q) = \widetilde{B}^s_{pq}(Q) \times \mathrm{Ext}^{\bar{r},u}_\Gamma \prod_{l=l_0}^{n-1} \prod_{|\alpha| \le r^l} \widetilde{B}^{s-\frac{n-l}{p}-|\alpha|}_{pq}(\Gamma_l) \tag{6.116}$$

and

$$F^s_{pq}(Q) = \widetilde{F}^s_{pq}(Q) \times \mathrm{Ext}^{\bar{r},u}_\Gamma \prod_{l=l_0}^{n-1} \prod_{|\alpha| \le r^l} \widetilde{B}^{s-\frac{n-l}{p}-|\alpha|}_{pp}(\Gamma_l) \tag{6.117}$$

(*complemented subspaces*).

Proof. We use Theorem 6.23 and an induction by dimension $l \ge l_0$. Let $l_0 \ge 1$. Then one has by Proposition 6.13 that

$$B^{s-\frac{n-l_0}{p}}_{pq}(\Gamma_{l_0}) = \widetilde{B}^{s-\frac{n-l_0}{p}}_{pq}(\Gamma_{l_0}). \tag{6.118}$$

Now one obtains by the above considerations with $r^{l_0} = 0$ that

$$\mathrm{Ext}^{r^{l_0},u}_{\Gamma_{l_0}} : B^{s-\frac{n-l_0}{p}}_{pq}(\Gamma_{l_0}) \hookrightarrow B^s_{pq}(Q). \tag{6.119}$$

If $l_0 = 0$ then one has to modify the above arguments according to Section 5.2.3 where we dealt with the one-dimensional case. This decomposes $B^s_{pq}(Q)$ as in (6.89), (6.90),

$$B^s_{pq}(Q) = \left\{ f \in B^s_{pq}(Q) : \operatorname{tr}^{r^{l_0}}_{\Gamma_{l_0}} f = 0 \right\} \times \operatorname{Ext}^{r^{l_0},u}_{\Gamma_{l_0}} B^{s-\frac{n-l_0}{p}}_{pq}(\Gamma_{l_0}) \tag{6.120}$$

into complemented subspaces. Let $l = l_0 + 1$. The trace space of the first factor on the right-hand side of (6.120) is now $\widetilde{B}^{s-\frac{n-l}{p}}_{pq}(\Gamma_l)$. But this observation ensures that one can apply the decomposition technique in connection with Theorem 6.23. Then one obtains the factor with $l = l_0 + 1$ on the right-hand side of (6.116). It remains a complemented subspace now with $\operatorname{tr}^{r^l}_{\Gamma_l} f = 0$ for $l = l_0$ and $l = l_0 + 1$. We remark again that according to the construction (6.60) the extensions at different values of l do not interfere. Iteration finally ends up with (6.116), (6.117) where one has to use the counterpart of (6.92) or Proposition 6.15 with $\Omega = Q$ both for B-spaces and F-spaces. This proves also (6.113)–(6.115). □

Remark 6.29. Beyond the somewhat cumbersome technicalities the proof relies on the following two remarkable facts:

1. With p, q, s as in (6.111) the boundary data of $f \in A^s_{pq}(Q)$ on different faces of Γ_l, but also for different Γ_l and $\Gamma_{l'}$ with $l \neq l'$ are totally decoupled.

2. The wavelet-friendly extension operator for a face of Γ_l respects this observation and does not interfere with the boundary data of other faces of Γ_l, and with the boundary data of faces $\Gamma_{l'}$ with $l \neq l'$.

6.1.6 Cubes and polyhedrons: Riesz bases

Theorem 6.28 brings us in the same position as in Chapter 5 where we asked for wavelet bases in spaces $A^s_{pq}(\Omega)$ in case of bounded C^∞ domains Ω. We rephrased in Theorem 6.7 some distinguished assertions obtained there now in terms of [oscillating]{interior} u-Riesz bases introduced in Definition 6.5. Let $a^s_{pq}(\mathbb{Z}^\Omega)$ with $a \in \{b, f\}$ be the sequence spaces according to (6.6) and Definition 6.1.

Theorem 6.30. *Let $\Omega = Q$ be the cube* (6.96) *in $\mathbb{R}^n$ with $n \geq 2$. Let $A^s_{pq}(\Omega)$ be the spaces according to Definition* 2.1 (i) *with*

$$1 \leq p < \infty, \quad s > \frac{1}{p}, \quad s - \frac{k}{p} \notin \mathbb{N}_0 \text{ for } k = 1, \dots, n, \tag{6.121}$$

$0 < q < \infty$ for B-spaces, $1 \leq q < \infty$ for F-spaces. Then $A^s_{pq}(\Omega)$ has for any $u \in \mathbb{N}$ with $u > s$ an oscillating u-Riesz basis.

Proof. By Theorem 6.7 (i) all spaces on the right-hand sides of (6.116), (6.117) have interior u-Riesz bases (where spaces on Γ_l are considered in $\mathbb{R}^l$). This is the point where one has to strengthen $0 < q < \infty$ in (6.111) for F-spaces by $1 \le q < \infty$ (ensuring $\sigma_{pq} = 0$ in (3.45)). We described in Proposition 5.34 and Theorem 5.27 for bounded C^∞ domains Ω how wavelet bases for $\tilde{A}^s_{pq}(\Omega)$ can be complemented by transferred wavelet bases for trace spaces such that one obtains wavelet bases for $A^s_{pq}(\Omega)$. This applies also to the above situation now based on (6.116), (6.117), resulting in the above theorem. □

Corollary 6.31. *Theorem 6.30 remains valid for bounded polyhedrons Ω in $\mathbb{R}^n$.*

Proof. The proof of Theorem 6.28 is based on the extension operators (6.59), (6.60), clipped together by (6.110). One does not need that the respective faces intersecting each other are orthogonal. It is sufficient to know that they intersect with non-zero angles. Similarly one can replace the perpendicular derivatives in (6.103) by oblique derivatives. This does not influence the arguments and results in the above assertion. □

6.1.7 Cellular domains: Riesz bases

After all these preparations we reach now the main goal of Section 6.1, the proof of Theorems 5.47 and 5.49. For sake of completeness we repeat in addition Theorem 5.43 rephrased in terms of u-Riesz bases.

Theorem 6.32. *Let Ω be a cellular domain in $\mathbb{R}^n$ with $n \ge 2$ according to Definition 6.9. Let $A^s_{pq}(\Omega)$ be the spaces as introduced in Definition 2.1* (i).

(i) *Then $A^s_{pq}(\Omega)$ with (6.121), $0 < q < \infty$ for B-spaces, $1 \le q < \infty$ for F-spaces, has for any $u \in \mathbb{N}$ with $u > s$ an oscillating u-Riesz basis according to Definition 6.5.*

(ii) *Let*

$$0 < p < \infty, \quad 0 < q < \infty, \quad -\infty < s < \min\left(\frac{1}{p}, \frac{n}{n-1}\right). \tag{6.122}$$

Then $A^s_{pq}(\Omega)$ has for any sufficiently large $u \in \mathbb{N}$,

$$u > u(s,p) \in \mathbb{N} \ \text{ for } B\text{-spaces}, \quad u > u(s,p,q) \in \mathbb{N} \ \text{ for } F\text{-spaces},$$

an interior u-Riesz basis according to Definition 6.5.

Proof. Part (ii) is covered by Theorem 5.43 and the above reformulations. As for part (i) we first remark that diffeomorphic maps of cubes onto corresponding domains generate isomorphic maps both for related spaces and for wavelet bases. The extension operators used in Theorem 6.28 transfer boundary spaces (and wavelets) from Γ_l to $\mathbb{R}^n$, in particular to all adjacent cells. Hence the constructions resulting in Theorem 6.28 apply simultaneously to all cells (diffeomorphic images of cubes and, more general, polyhedrons). Then one obtains the theorem as in Theorem 6.30 and Corollary 6.31. □

Theorem 6.33. *Let $n \geq 2$. Theorem 6.32 remains valid for any bounded C^∞ domain in $\mathbb{R}^n$ according to Definition 3.4* (iii).

Proof. This follows from Theorem 6.32 and Proposition 6.11 (i). □

Remark 6.34. In Definition 6.5 we introduced [oscillating]{interior} u-Riesz bases and mentioned in Remark 6.6 that there is an obvious counterpart in terms of u-Riesz frames. According to Theorem 5.51 one has u-Riesz frames for all spaces $A^s_{pq}(\Omega)$ in bounded Lipschitz domains Ω without any exceptional values p and s. Under the restrictions of Theorem 5.27, in particular $s - \frac{1}{p} \notin \mathbb{N}_0$, the admitted spaces $A^s_{pq}(\Omega)$ have oscillating u-wavelet frames. The step from frames to bases requires by our method additional assumptions of type (6.121). The restriction $s - \frac{1}{p} \notin \mathbb{N}_0$ is natural. We discuss this point in Sections 6.2.2–6.2.4 below. The situation might be different in case of the additional restrictions $s - \frac{k}{p} \notin \mathbb{N}_0$ with $2 \leq k \in \mathbb{N}$. For some bounded C^∞ domains Ω in $\mathbb{R}^n$ with $n \geq 3$ we have the restrictions (5.215). If Ω is a cube or a cellular domain then one has in Theorems 6.30, 6.32 even the restrictions (6.121). We discussed this situation in Section 5.3.1. Let

$$H^s_p(\Omega) = F^s_{p,2}(\Omega), \quad 1 < p < \infty, \quad s \in \mathbb{R}, \tag{6.123}$$

be the usual Sobolev spaces (the restrictions of the spaces $H^s_p(\mathbb{R}^n)$ in (1.17)–(1.19) to Ω). Let $\mathbb{M}^3$ be the torus of revolution in $\mathbb{R}^3$ as in (5.197) and let $\mathbb{B}^3$ be the unit ball in $\mathbb{R}^3$ as in (5.198). Then it follows from Theorems 5.38, 6.32, 6.33 and Remark 5.36 that one has oscillating u-Riesz bases in

$$H^s_p(\mathbb{M}^3) \quad \text{if } 1 < p < \infty, 0 < s - \tfrac{1}{p} \notin \mathbb{N}, \tag{6.124}$$

in

$$H^s_p(\mathbb{B}^3) \quad \text{if } 1 < p < \infty, 0 < s - \tfrac{1}{p} \notin \mathbb{N}, s - \tfrac{2}{p} \notin \mathbb{N}_0, \tag{6.125}$$

and in

$$H^s_p(\Omega) \quad \text{if } 1 < p < \infty, 0 < s - \tfrac{1}{p} \notin \mathbb{N}, s - \tfrac{2}{p} \notin \mathbb{N}_0, s - \tfrac{3}{p} \notin \mathbb{N}_0, \tag{6.126}$$

for cellular (in particular bounded C^∞) domains Ω in $\mathbb{R}^3$. But it is unlikely that the additional exceptional values in (6.125) or in (6.126) are naturally related to the different topologies of $\partial\mathbb{M}^3$, $\partial\mathbb{B}^3$ or $\partial\Omega$. The situation is especially curious if $p = 2$. Then it is unclear (by our method) whether the very classical Sobolev spaces

$$W^k_2(\mathbb{B}^3), \quad k \in \mathbb{N}, \tag{6.127}$$

have oscillating u-Riesz bases.

6.2 Existence and non-existence of wavelet frames and bases

6.2.1 The role of duality, the spaces $\mathbf{B}^s_{pq}(\mathbb{R}^n)$

Recall that $\{b_j\}_{j=1}^\infty \subset B$ in a separable complex quasi-Banach space B is called a *basis* if any $b \in B$ can be uniquely represented as

$$b = \sum_{j=1}^\infty \lambda_j \, b_j, \quad \lambda_j \in \mathbb{C} \quad \text{(convergence in } B). \tag{6.128}$$

Of course $\lambda_j(b)$ is linear in b. A basis is called a *Schauder basis* if $\lambda_j(\cdot) \in B'$ are linear and continuous functionals on B. It is one of the very first observations of Banach space theory by Banach himself, [Ban32], that any basis in a Banach space is a Schauder basis, an early application of the Closed Graph Theorem. Recent proofs may be found in [AlK06], Theorem 1.1.3, p. 3, and [Woj91], Corollary, p. 38. However this observation can be extended to quasi-Banach spaces.

Proposition 6.35. *Any basis in a separable complex quasi-Banach space B is a Schauder basis.*

Proof. For any quasi-metric ϱ on a set X there is a number ε_0 with $0 < \varepsilon_0 \le 1$ such that ϱ^ε for ε with $0 < \varepsilon \le \varepsilon_0$ is equivalent to a metric. A proof may be found in [Hei01], Proposition 14.5. One may also consult [T06], Section 1.17.4. As a consequence, any quasi-Banach is a Fréchet space (also called an F-space, but should not to be mixed with our occasional abbreviation of F^s_{pq}-spaces as F-spaces) for which the Closed Graph Theorem remains to be valid, [Woj91], pp. 3–4. One may also consult [Yos80], I.9, II.6, pp. 52, 79. Now one can carry over the proof in [AlK06], Theorem 1.1.3, pp. 3, 4, from Banach spaces to quasi-Banach spaces. □

Remark 6.36. Inserting $b = b_k$ in (6.128) the uniqueness of the representation ensures the existence of a dual system $\{b'_j\}_{j=1}^\infty \subset B'$ with

$$b'_j\big(b_k\big) = \delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \ne k. \end{cases} \tag{6.129}$$

Hence the existence of a basis in a quasi-Banach space B requires that it has a sufficiently rich dual B'. This is always the case for all spaces $A^s_{pq}(\mathbb{R}^n)$ and $A^s_{pq}(\Omega)$ considered in the preceding chapters. But it is not the case, as we shall see, for other function spaces of interest. Even worse, one has the same negative outcome if one asks for frames instead of bases. Similarly as in Remark 5.11 we call $\{b_j\}_{j=1}^\infty \subset B$ a (stable) *frame* if there is a quasi-Banach space Λ of $\{\lambda_j \in \mathbb{C} : j \in \mathbb{N}\}$ (sequence space) with the following properties:

1. (*Stability*) Any $b \in B$ can be represented as

$$b = \sum_{j=1}^\infty \lambda_j b_j, \quad \lambda = \{\lambda_j\} \in \Lambda \quad \text{(convergence in } B), \tag{6.130}$$

with

$$\|b\,|B\| \sim \inf \|\lambda\,|\Lambda\| \tag{6.131}$$

where the infimum is taken over all representations (6.130).

2. (*Optimality*) There is a sequence $\lambda = \{\lambda_j\} \subset B'$ with

$$b = \sum_{j=1}^{\infty} \lambda_j(b)\, b_j, \quad \|\lambda(b)\,|\Lambda\| \sim \|b\,|B\|, \tag{6.132}$$

convergence in B.

Let $\Delta_h^m f$ be the differences in $\mathbb{R}^n$ according to (1.21). Let

$$\mathbf{B}^s_{pq}(\mathbb{R}^n) = \big\{ f \in L_p(\mathbb{R}^n) : \ \|f\,|\mathbf{B}^s_{pq}(\mathbb{R}^n)\|_m < \infty \big\} \tag{6.133}$$

where

$$0 < p < \infty, \quad 0 < q < \infty, \quad 0 < s < m \in \mathbb{N}, \tag{6.134}$$

and

$$\|f\,|\mathbf{B}^s_{pq}(\mathbb{R}^n)\|_m = \|f\,|L_p(\mathbb{R}^n)\| + \Big(\int_{\mathbb{R}^n} |h|^{-sq}\, \|\Delta_h^m f\,|L_p(\mathbb{R}^n)\|^q \,\frac{\mathrm{d}h}{|h|^n} \Big)^{1/q}. \tag{6.135}$$

We dealt in [T06], Chapter 9, with these spaces, where one finds also references to the original papers. This will not be repeated here. One can replace the integration over $h \in \mathbb{R}^n$ in the second term on the right-hand side of (6.135) by $|h| \le 1$. Then one obtains (1.23), and by (1.25) that

$$B^s_{pq}(\mathbb{R}^n) = \mathbf{B}^s_{pq}(\mathbb{R}^n), \quad 0 < p, q < \infty, \ s > \sigma_p = n\big(\tfrac{1}{p} - 1\big)_+, \tag{6.136}$$

(appropriately interpreted as subspace of $S'(\mathbb{R}^n)$ and of $L_1^{\mathrm{loc}}(\mathbb{R}^n)$, justified by the embedding $\mathbf{B}^s_{pq}(\mathbb{R}^n) \subset L_1^{\mathrm{loc}}(\mathbb{R}^n)$). By Theorem 1.20 all spaces covered by (6.136) have wavelet bases. If $0 < s < \sigma_p$ then the spaces $B^s_{pq}(\mathbb{R}^n)$ and $\mathbf{B}^s_{pq}(\mathbb{R}^n)$ do not coincide any longer (they are even not comparable). As usual, B' is the dual of the quasi-Banach space B.

Theorem 6.37. *Let*

$$0 < p < 1, \quad 0 < q < \infty, \quad 0 < s < \sigma_p = n\Big(\frac{1}{p} - 1\Big). \tag{6.137}$$

Then

$$\mathbf{B}^s_{pq}(\mathbb{R}^n)' = \{0\} \tag{6.138}$$

for the dual space of $\mathbf{B}^s_{pq}(\mathbb{R}^n)$. *Furthermore,*

$$L_p(\mathbb{R}^n)' = \{0\}. \tag{6.139}$$

Proof. Let $\varphi \in D(\mathbb{R}^n)$ and $\varphi_j(x) = \varphi\big(2^j x\big)$ where $j \in \mathbb{N}$. Then one obtains by (6.135) that

$$\begin{aligned} &\|\varphi_j \,|\mathbf{B}^s_{pq}(\mathbb{R}^n)\|_m \\ &\quad = 2^{-j\frac{n}{p}}\, \|\varphi\,|L_p(\mathbb{R}^n)\| + 2^{j(s-\frac{n}{p})} \bigg(\int_{\mathbb{R}^n} |h|^{-sq}\, \|\Delta^m_h \varphi\,|L_p(\mathbb{R}^n)\|^q \,\frac{\mathrm{d}h}{|h|^n} \bigg)^{1/q} \\ &\quad \le 2^{j(s-\frac{n}{p})}\, \|\varphi\,|\mathbf{B}^s_{pq}(\mathbb{R}^n)\|_m . \end{aligned} \tag{6.140}$$

We assume that $\varphi \in D(\mathbb{R}^n)$, say, with $\varphi \ge 0$, generates a resolution of unity,

$$1 = \sum_{k\in\mathbb{Z}^n} \varphi(x-k), \qquad x \in \mathbb{R}^n. \tag{6.141}$$

Let $g \in \mathbf{B}^s_{pq}(\mathbb{R}^n)'$ and $\psi \in D(\mathbb{R}^n)$. Then

$$\begin{aligned} |g(\psi)| &\le \sum_{|k|\le K} |g(\psi\,\varphi(2^j\cdot -k))| \\ &\le \|g\| \sum_{|k|\le K} \|\psi\,\varphi(2^j\cdot -k)\,|\mathbf{B}^s_{pq}(\mathbb{R}^n)\|_m \\ &\le c\, 2^{jn}\, 2^{j(s-\frac{n}{p})} = c\, 2^{j(s+n-\frac{n}{p})}, \end{aligned} \tag{6.142}$$

where we used $K \sim 2^{jn}$, (6.140) and that ψ is a pointwise multiplier in $\mathbf{B}^s_{pq}(\mathbb{R}^n)$. This can be checked by direct calculation or by real interpolation between $L_p(\mathbb{R}^n)$ and the spaces $B^\sigma_{pq}(\mathbb{R}^n)$ with $\sigma > \sigma_p$ as indicated in [T01], pp. 373–74. The constant c in (6.142) depends on g and ψ, but not on j. Since $s < \frac{n}{p} - n$ one obtains by $j \to \infty$ that $g(\psi) = 0$. Again by real interpolation as indicated above or by direct calculation it follows that $D(\mathbb{R}^n)$ is dense in $\mathbf{B}^s_{pq}(\mathbb{R}^n)$. Hence $g = 0$. This proves (6.138). One obtains (6.139) by the same arguments. □

Corollary 6.38. *Neither the spaces $L_p(\mathbb{R}^n)$ nor the spaces $\mathbf{B}^s_{pq}(\mathbb{R}^n)$ with (6.137) have bases or frames with (6.130)–(6.132).*

Proof. This follows from Theorem 6.37, Proposition 6.35 and Remark 6.36. □

Remark 6.39. In [T06], Chapter 9, we dealt also with spaces $\mathbf{F}^s_{pq}(\mathbb{R}^n)$ and obtained under some restrictions of the parameters p and q,

$$\mathbf{B}^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow \mathbf{F}^s_{pq}(\mathbb{R}^n) \hookrightarrow \mathbf{B}^s_{p,\max(p,q)}(\mathbb{R}^n) \tag{6.143}$$

similarly as for the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. Then one obtains from (6.138) and (6.143) that

$$\mathbf{F}^s_{pq}(\mathbb{R}^n)' = \{0\}. \tag{6.144}$$

In particular these spaces do not have bases or frames with (6.130)–(6.132). However one has a remarkable substitute. There are constructive building blocks, called *quarks*, both for $\mathbf{B}^s_{pq}(\mathbb{R}^n)$ and $\mathbf{F}^s_{pq}(\mathbb{R}^n)$ with expansions of type (6.130), (6.131). We refer for details and further information to [T06], Section 9.2, and the recent papers [HaS08], [Sch08], [Sch09].

6.2.2 The non-existence of Riesz frames in exceptional spaces

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \ge 2$ according to Definition 3.4 (iii) or a bounded interval in $\mathbb{R}$ if $n = 1$. Let

$$\Phi = \left\{ \Phi^j_l : j \in \mathbb{N}_0;\ l = 1, \ldots, N_j \right\} \subset C^u(\Omega) \tag{6.145}$$

be an [oscillating]{interior} u-wavelet system as introduced in Definition 6.3. In Definition 6.5 we said what is meant by related [oscillating]{interior} u-Riesz bases. Let $\hat{A}^s_{pq}(\Omega)$ be either $A^s_{pq}(\Omega)$ or $\tilde{A}^s_{pq}(\Omega)$ according to Definition 2.1, or $\mathring{A}^s_{pq}(\Omega)$ as in Definition 5.17 (i). Let $a^s_{pq}(\mathbb{Z}^\Omega)$ be as in (6.6) and Definition 6.1. In good agreement with the comments in Remarks 6.6 and 6.36 we call the [oscillating]{interior} u-wavelet system Φ in (6.145) an [oscillating]{interior} *u-Riesz frame* for

$$\hat{A}^s_{pq}(\Omega) \quad \text{with } 0 < p < \infty,\ 0 < q < \infty,\ s \in \mathbb{R},$$

if it has the following properties:

1. An element $f \in D'(\Omega)$ belongs to $\hat{A}^s_{pq}(\Omega)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda^j_l \, 2^{-jn/2} \, \Phi^j_l, \quad \lambda \in a^s_{pq}(\mathbb{Z}^\Omega), \tag{6.146}$$

   unconditional convergence being in $\hat{A}^s_{pq}(\Omega)$, and

$$\| f \,|\hat{A}^s_{pq}(\Omega)\| \sim \inf \|\lambda \,|a^s_{pq}(\mathbb{Z}^\Omega)\|, \tag{6.147}$$

   where the infimum is taken over all admissible representations (6.146).

2. Any $f \in \hat{A}^s_{pq}(\Omega)$ can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda^j_l(f) \, 2^{-jn/2} \, \Phi^j_l \tag{6.148}$$

   where $\lambda^j_l(\cdot) \in \hat{A}^s_{pq}(\Omega)'$ are linear and continuous functionals on $\hat{A}^s_{pq}(\Omega)$ and

$$\| f \,|\hat{A}^s_{pq}(\Omega)\| \sim \|\lambda(f) \,|a^s_{pq}(\mathbb{Z}^\Omega)\|. \tag{6.149}$$

This is the direct counterpart of Definition 6.5. Theorem 5.27 ensures the existence of oscillating u-Riesz frames in some spaces $A^s_{pq}(\Omega)$ excluding the exceptional spaces with $s - \frac{1}{p} \in \mathbb{N}_0$. In this Section 6.2.2 and in the following Section 6.2.3 we have a closer look at these exceptional spaces.

A bounded Lipschitz domain Ω is E-thick. This follows from Definitions 3.1, 3.4 and Proposition 3.8. Then one has by Theorem 3.13 that the spaces

$$\tilde{A}^s_{pq}(\Omega), \quad 0 < p, q < \infty, \quad s > \begin{cases} \sigma_p, & B\text{-spaces}, \\ \sigma_{pq}, & F\text{-spaces}, \end{cases} \tag{6.150}$$

have interior u-Riesz bases according to Definition 6.5 and hence also interior u-Riesz frames with $u > s$. The isomorphic map onto $a^s_{pq}(\mathbb{Z}_\Omega)$, now identified with $a^s_{pq}(\mathbb{Z}^\Omega)$, in Theorem 3.13 and a subsequent mollification show that $D(\Omega)$ is dense in the spaces $\tilde{A}^s_{pq}(\Omega)$ with (6.150). (Of course, extending Definition 6.5 and the above definition to u-Riesz frames one has the same assertions for E-thick domains.)

Proposition 6.40. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition* 3.4 (iii) *or a bounded interval in $\mathbb{R}$ if $n = 1$. Let*

$$0 < p, q < \infty, \quad s > \begin{cases} \sigma_p, & B\text{-spaces}, \\ \sigma_{pq}, & F\text{-spaces}. \end{cases} \tag{6.151}$$

Let $s < u \in \mathbb{N}$. Then $\mathring{A}^s_{pq}(\Omega)$ has an interior u-Riesz frame if, and only if,

$$\mathring{A}^s_{pq}(\Omega) = \tilde{A}^s_{pq}(\Omega). \tag{6.152}$$

Proof. By the above considerations it follows from (6.152) that $\mathring{A}^s_{pq}(\Omega)$ has an interior u-Riesz basis and hence an interior u-Riesz frame. Conversely, if $\mathring{A}^s_{pq}(\Omega)$ has an interior u-Riesz frame then any $f \in D(\Omega)$ can be represented by (6.148), (6.149). Extended to $\mathbb{R}^n$ by zero outside of Ω one has an atomic decomposition of $f \in D(\mathbb{R}^n)$ in $A^s_{pq}(\mathbb{R}^n)$ according to Theorem 1.7 (no moment conditions are needed). Hence,

$$\| f \,|\tilde{A}^s_{pq}(\Omega)\| = \| f \,|A^s_{pq}(\mathbb{R}^n)\| \leq c\, \| f \,|A^s_{pq}(\Omega)\|. \tag{6.153}$$

The converse of (6.153) follows from (2.5). Since $D(\Omega)$ is dense both in $\mathring{A}^s_{pq}(\Omega)$ and $\tilde{A}^s_{pq}(\Omega)$ one obtains (6.152) from

$$\| f \,|\tilde{A}^s_{pq}(\Omega)\| \sim \| f \,|A^s_{pq}(\Omega)\|, \quad f \in D(\Omega). \tag{6.154}$$

□

Remark 6.41. For cellular domains Ω (and hence for bounded C^∞ domains) we have so far Propositions 6.13, 6.15, ensuring (6.152) under the given restrictions for the parameters p, q, s. According to Remark 5.20 and the references given there one has for bounded C^∞ domains Ω,

$$\mathring{A}^s_{pq}(\Omega) = \tilde{A}^s_{pq}(\Omega) \quad \text{if } 0 < p, q < \infty,\ s > \sigma_p,\ s - \tfrac{1}{p} \notin \mathbb{N}_0. \tag{6.155}$$

Hence by the remarks after (6.150) the spaces $\mathring{A}^s_{pq}(\Omega)$ with (6.151) and $s - \frac{1}{p} \notin \mathbb{N}_0$ have interior u-Riesz frames (even interior u-Riesz bases).

Theorem 6.42. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition* 3.4 (iii) *or a bounded interval in $\mathbb{R}$ if $n = 1$. Let $\mathring{A}^s_{pq}(\Omega)$ be the spaces as introduced in Definition* 5.17 (i). *Let*

$$0 < p, q < \infty, \quad s - \frac{1}{p} = r \in \mathbb{N}_0 \quad \text{and} \quad s < u \in \mathbb{N}. \tag{6.156}$$

Then neither

$$\mathring{B}^s_{pq}(\Omega) \quad \text{with } \min(1,p) < q < \infty, \; s > \sigma_p, \tag{6.157}$$

nor

$$\mathring{F}^s_{pq}(\Omega) \quad \text{with } p > 1, \; 0 < q < \infty, \; s > \sigma_q. \tag{6.158}$$

have an interior u-Riesz frame.

Proof. The case $r = 0$ follows from (5.122), (5.126) and Proposition 6.40. The extension of this assertion to $r \in \mathbb{N}$ is covered by (5.128). □

Remark 6.43. The above theorem makes clear that the restrictions for s in Theorems 5.21, 5.27 are natural.

6.2.3 Reinforced spaces

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$. Then one has u-Riesz frames (= u-wavelet frames) in the spaces $A^s_{pq}(\Omega)$ covered by Theorem 5.27. This excludes spaces with $s - \frac{1}{p} \in \mathbb{N}_0$. Theorem 6.42 shows that the situation in these exceptional spaces is rather peculiar. Let

$$d(x) = \operatorname{dist}(x, \Gamma) = \inf\{|x - \gamma| : \gamma \in \Gamma\}, \quad x \in \Omega, \tag{6.159}$$

be the distance to the boundary $\Gamma = \partial\Omega$. Then one has by (5.122), (5.126) and [T01], Theorem 5.10. p. 54, for $1 < p < \infty$, $1 \le q < \infty$,

$$\mathring{F}^{1/p}_{pq}(\Omega) = F^{1/p}_{pq}(\Omega) \neq \widetilde{F}^{1/p}_{pq}(\Omega) = \big\{ f \in F^{1/p}_{pq}(\Omega) : d^{-1/p} f \in L_p(\Omega) \big\}. \tag{6.160}$$

Although $D(\Omega)$ is dense in $F^{1/p}_{pq}(\Omega)$ one obtains by Theorem 6.42 that $F^{1/p}_{pq}(\Omega)$ has no interior u-Riesz frame or interior u-Riesz basis. The Hardy inequality

$$\int_\Omega |f(x)|^p \, \frac{\mathrm{d}x}{d(x)} \le c \, \|f \,|\widetilde{F}^{1/p}_{pq}(\Omega)\|^p, \quad f \in \widetilde{F}^{1/p}_{pq}(\Omega), \tag{6.161}$$

is a special case of (5.124). The theory of envelopes as developed in [T01], Chapter II, and in [Har07] produces the following *sharp Hardy inequality* for the spaces $F^{1/p}_{pq}(\Omega)$. Let again $1 < p < \infty$, $1 \le q < \infty$, and let $\varkappa(t)$ with $t \ge 0$ be a positive monotonically decreasing function. Then one has for some $c > 0$,

$$\int_\Omega \left| \frac{\varkappa\big(d(x)\big) \, f(x)}{1 + |\log d(x)|} \right|^p \frac{\mathrm{d}x}{d(x)} \le c \, \big\|f \,|F^{1/p}_{pq}(\Omega)\big\|^p, \quad f \in F^{1/p}_{pq}(\Omega), \tag{6.162}$$

if, and only if, $\varkappa$ is bounded. We refer to [T01], Corollary 16.7, Remark 16.8, p. 242. Hence one obtains the sharp Hardy inequality

$$\int_\Omega \left| \frac{f(x)}{1 + |\log d(x)|} \right|^p \frac{\mathrm{d}x}{d(x)} \le c \, \big\|f \,|F^{1/p}_{pq}(\Omega)\big\|^p, \quad f \in F^{1/p}_{pq}(\Omega). \tag{6.163}$$

The sharp Hardy inequalities (6.161), (6.163) show the difference between these two spaces. In contrast to this observation one has for $1 \le p, q < \infty$, $0 < s < 1/p$, that

$$F^s_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega), \qquad \int_\Omega |f(x)|^p \, \frac{\mathrm{d}x}{d^{sp}(x)} \le c \, \| f \, | F^s_{pq}(\Omega) \|^p \tag{6.164}$$

for some $c > 0$ and all $f \in F^s_{pq}(\Omega)$. These are special cases of Proposition 5.19 and (5.124). This suggests to reinforce the spaces $F^s_{pq}(\Omega)$ with $s - \frac{1}{p} \in \mathbb{N}_0$ as follows. Let $\nu = \nu(\gamma)$ with $\gamma \in \Gamma = \partial\Omega$ be the outer normal at the boundary Γ of the above bounded C^∞ domain Ω in $\mathbb{R}^n$ (bounded interval if $n = 1$), naturally extended to

$$\Omega_\varepsilon = \{x \in \Omega, \ d(x) < \varepsilon\}, \qquad \text{with } \varepsilon > 0 \text{ sufficiently small.} \tag{6.165}$$

Definition 6.44. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ with $n \ge 2$ according to Definition 3.4 (iii) or a bounded interval in $\mathbb{R}$ if $n = 1$. Let $1 \le p, q < \infty$ and $s > \frac{1}{p} - 1$. Then

$$F^{s,\mathrm{rinf}}_{pq}(\Omega) = \begin{cases} F^s_{pq}(\Omega) & \text{if } s - \frac{1}{p} \notin \mathbb{N}_0, \\ \big\{ f \in F^s_{pq}(\Omega) : \, d^{-1/p} \frac{\partial^r f}{\partial \nu^r} \in L_p(\Omega_\varepsilon) \big\} & \text{if } s - \frac{1}{p} = r \in \mathbb{N}_0. \end{cases} \tag{6.166}$$

Remark 6.45. If $s - \frac{1}{p} = r \in \mathbb{N}_0$ then $F^{s,\mathrm{rinf}}_{pq}(\Omega)$ is normed by

$$\| f \, | F^{s,\mathrm{rinf}}_{pq}(\Omega) \| = \| f \, | F^s_{pq}(\Omega) \| + \bigg(\int_{\Omega_\varepsilon} \Big| \frac{\partial^r f(x)}{\partial \nu^r} \Big|^p \, \frac{\mathrm{d}x}{d(x)} \bigg)^{1/p}. \tag{6.167}$$

If $r = 0$ then $\frac{\partial^r f}{\partial \nu^r} = f$ and one obtains by [T01], Theorem 5.10, p. 54, incorporating $p \ge 1$ in the right-and side of (6.160),

$$F^{1/p,\mathrm{rinf}}_{pq}(\Omega) = \widetilde{F}^{1/p}_{pq}(\Omega), \qquad 1 \le p, q < \infty. \tag{6.168}$$

Now one can extend the F-part of Theorem 5.27 to the above spaces with $s = r + \frac{1}{p}$, where we now use the notation of an oscillating u-Riesz frame as introduced at the beginning of Section 6.2.2.

Theorem 6.46. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ with $n \ge 2$ according to Definition* 3.4 (iii) *or a bounded interval in $\mathbb{R}$ if $n = 1$. Let $r \in \mathbb{N}_0$ and $u \in \mathbb{N}$ with $r < u$. Then there is a common oscillating u-Riesz frame according to* (6.146)–(6.149) *for all spaces $F^{s,\mathrm{rinf}}_{pq}(\Omega)$ with*

$$1 \le p < \infty, \quad 1 \le q < \infty, \quad \frac{1}{p} - 1 < s - r \le \frac{1}{p}. \tag{6.169}$$

Proof. Step 1. In case of $s - \frac{1}{p} \notin \mathbb{N}_0$ the above assertion is covered by Theorem 5.27. It remains to deal with the exceptional spaces $F^{s,\mathrm{rinf}}_{pq}(\Omega)$ where $s = \frac{1}{p} + r$. Let $r = 0$

and $u \in \mathbb{N}$. Then one obtains the desired assertion (as in Step 1 of the proof of Theorem 5.27) from Theorem 3.23 and (6.168).

Step 2. Let $r \in \mathbb{N}$. We wish to show that the common u-Riesz frame for the F-spaces with (5.142) is also a u-Riesz frame for $F^{s,\mathrm{rinf}}_{pq}(\Omega)$ with $s = \frac{1}{p} + r$. We begin with a preparation. We claim that

$$\widetilde{F}^s_{pq}(\Omega) = \left\{ f \in F^{s,\mathrm{rinf}}_{pq}(\Omega) : \ \mathrm{tr}^{r-1}_\Gamma f = 0 \right\}, \tag{6.170}$$

where $\mathrm{tr}^{r-1}_\Gamma f$ has the same meaning as in (5.59) with $r-1$ in place of r. Let $f \in \widetilde{F}^s_{pq}(\Omega)$. Since $D(\Omega)$ is dense in $\widetilde{F}^s_{pq}(\Omega)$ and

$$\widetilde{F}^s_{pq}(\Omega) \hookrightarrow F^s_{pq}(\Omega) \tag{6.171}$$

one obtains by (5.62), (5.63) that $\mathrm{tr}^{r-1}_\Gamma f = 0$. By Definition 2.1 (ii) one has

$$D^\alpha f \in \widetilde{F}^{1/p}_{pq}(\Omega) \quad \text{if } |\alpha| = r \text{ and } f \in \widetilde{F}^s_{pq}(\Omega). \tag{6.172}$$

By (6.166) and (6.160) it follows that $f \in \widetilde{F}^s_{pq}(\Omega)$ with $s = \frac{1}{p} + r$ is an element of the right-hand side of (6.170). We prove the converse. Let $n = 1$, hence $\Omega = I$ is a bounded interval, say, $I = (0,1)$. It is sufficient to look what happens at $x = 0$. If f is an element of the spaces on the right-hand side of (6.170), say with $f(x) = 0$ if $x > 1/2$, then

$$f \in F^s_{pq}(I), \qquad \begin{cases} f^{(l)}(0) = 0 & \text{if } l = 0, \dots, r-1, \\ \int_0^\infty |f^{(l)}(x)|^p \, \frac{dx}{x} < \infty & \text{if } l = r. \end{cases} \tag{6.173}$$

We extend f by zero to $(-\infty, 0)$. This is also an extension of the distributional derivatives $f^{(l)}$ with $l = 0, \dots, r$. By the last equality in (6.160) (which applies also to $p = 1$, [T01], p. 54) it follows from (6.173) that $f \in \widetilde{F}^s_{pq}(I)$. This proves (6.170) in case of $n = 1$. Let $n \geq 2$. Localisation and diffeomorphic maps show that we may assume that f is supported near the origin and that a piece of the hyper-plane $\{x_1 = 0\}$ is part of the boundary Γ, in particular $\frac{\partial}{\partial \nu} = \frac{\partial}{\partial x_1}$. Then one obtains the desired assertion from the one-dimensional case and the Fubini property for the F-spaces according to [T01], Theorem 4.4, p. 36.

Step 3. After this preparation we extend now the construction in Theorem 5.27 for p, q, s with (5.142) to $F^{s,\mathrm{rinf}}_{pq}(\Omega)$ with $s = \frac{1}{p} + r$ where $r \in \mathbb{N}$. Let $\mathrm{Ext}^{r-1,u}_\Gamma$ be the same extension operator as in (5.152). Then one has by Theorem 5.14

$$\mathrm{ext}^{r-1,u}_\Gamma = \mathrm{re}_\Omega \circ \mathrm{Ext}^{r-1,u}_\Gamma : \prod_{k=0}^{r-1} B^{r-k}_{pp}(\Gamma) \hookrightarrow F^s_{pq}(\Omega). \tag{6.174}$$

Let g be as in (5.152) with the building blocks $\Phi^{j,k}_l(\gamma)$ in (5.153). From the explicit form of $\Phi^{j,k}_l(\gamma)$ and (5.72) it follows that

$$\frac{\partial^r}{\partial \gamma_n^r} \Phi^{j,k}_l(\gamma) = 0 \ \text{ near } \Gamma; \quad k = 0, \dots, r-1. \tag{6.175}$$

In particular, $\frac{\partial^r}{\partial \gamma_n^r} g$ is an atomic decomposition in $F^{1/p}_{pq}(\mathbb{R}^n)$. Hence

$$\frac{\partial^r g}{\partial \nu^r} \in \widetilde{F}^{1/p}_{pq}(\Omega). \tag{6.176}$$

Using once more the right-hand side of (6.160) (again extended to $p = 1$) one obtains by (6.167) and (6.174) that

$$\mathrm{ext}^{r-1,u}_{\Gamma} : \prod_{k=0}^{r-1} B^{r-k}_{pp}(\Gamma) \hookrightarrow F^{s,\mathrm{rinf}}_{pq}(\Omega). \tag{6.177}$$

This is the decisive observation. Then (5.62) can be strengthened by

$$\mathrm{tr}^{r-1}_{\Gamma} F^{s,\mathrm{rinf}}_{pq}(\Omega) = \prod_{k=0}^{r-1} B^{r-k}_{pp}(\Gamma). \tag{6.178}$$

But now one is in the same position as in Corollary 5.16 and (5.113) (with $r-1$ in place of r). In particular, one obtains by (6.170) that

$$Q^{r-1,u} F^{s,\mathrm{rinf}}_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega). \tag{6.179}$$

As a counterpart of (5.117) one has the decomposition

$$F^{s,\mathrm{rinf}}_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega) \times \mathrm{ext}^{r-1,u}_{\Gamma} \prod_{k=0}^{r-1} B^{r-k}_{pp}(\Gamma). \tag{6.180}$$

Then one obtains the above theorem by the same arguments as in the proof of Theorem 5.27. □

Remark 6.47. We removed the restrictions $s - \frac{1}{p} \notin \mathbb{N}_0$ in Theorem 5.27 for the F-spaces at the expense of the reinforced spaces in (6.166). Recall that this covers in particular the reinforced Sobolev spaces

$$H^{s,\mathrm{rinf}}_p(\Omega) = F^{s,\mathrm{rinf}}_{p,2}(\Omega), \quad 1 < p < \infty, \ s - \tfrac{1}{p} \in \mathbb{N}_0. \tag{6.181}$$

One may ask for u-Riesz bases according to Definition 6.5 naturally extended to $F^{s,\mathrm{rinf}}_{pq}(\Omega)$. One can again rely on the obviously modified Proposition 5.34.

Corollary 6.48. *Let Ω be either a bounded interval on $\mathbb{R}$ $(n = 1)$ or a bounded planar C^∞ domain in $\mathbb{R}^2$ $(n = 2)$. Let $r \in \mathbb{N}_0$ and $u \in \mathbb{N}$ with $r < u$. Then there is a common oscillating u-Riesz basis for all spaces $F^{s,\mathrm{rinf}}_{pq}(\Omega)$ with*

$$1 \le p < \infty, \quad 1 \le q < \infty, \quad \frac{1}{p} - 1 < s - r \le \frac{1}{p}. \tag{6.182}$$

Proof. One can use the same arguments as in the proof of Theorem 5.35 based on (6.180). □

6.2.4 A proposal

In Section 5.3 and also in Sections 6.1.6, 6.1.7 we dealt with u-Riesz bases for spaces in higher dimensions based on induction by dimensions. This produces additional exceptional values of s as in Theorems 5.38, 6.30, 6.32. We discussed these effects in Section 5.3.1 and in Remark 6.34. The question arises of whether one can remove these restrictions by reinforcing the original spaces in a similar way as in (6.167). Now $d(x)$ may be the distance to some edges, cutting faces or cutting lines of lower dimensions. A corresponding theory has not yet been worked out. But we illustrate these admittedly cryptical comments in a somewhat sketchy way by an example. Let

$$H^1(\mathbb{R}^n) = W_2^1(\mathbb{R}^n) = F^1_{2,2}(\mathbb{R}^n) \tag{6.183}$$

be the distinguished classical Sobolev space in $\mathbb{R}^n$, normed by

$$\|f\,|H^1(\mathbb{R}^n)\| = \sum_{|\alpha|\le 1} \|D^\alpha f\,|L_2(\mathbb{R}^n)\|. \tag{6.184}$$

The space $H^1(\mathbb{R}^2)$ in the plane $\mathbb{R}^2$ is exceptional in the context of the theory of envelopes as developed in [T01], Chapter II, [Har07]. Similarly as in (6.162), (6.163) one has the following *sharp Hardy inequality*. Let, as there, $\varkappa(t)$ with $t \ge 0$ be a positive monotonically decreasing function. Then one has for some $c > 0$,

$$\int_{|x|\le 1} \left| \frac{\varkappa(|x|)\, f(x)}{1 + \big|\log|x|\big|} \right|^2 \frac{\mathrm{d}x}{|x|^2} \le c\, \|f\,|H^1(\mathbb{R}^2)\|^2 \tag{6.185}$$

if, and only if, $\varkappa$ is bounded. This is a special case of [T01], Theorem 16.2, p. 237. On the other hand, the related refined localisation space $H^{1,\mathrm{rloc}}(\mathbb{R}^2 \setminus \{0\})$ according to Corollary 2.20 can be equivalently normed by

$$\|f\,|H^{1,\mathrm{rloc}}(\mathbb{R}^2\setminus\{0\})\| = \|f\,|H^1(\mathbb{R}^2)\| + \left(\int_{\mathbb{R}^2} |f(x)|^2 \frac{\mathrm{d}x}{|x|^2} \right)^{1/2}. \tag{6.186}$$

The situation is similar as in (6.162)–(6.164). Let $n \ge 2$. The unit cube and the unit ball are denoted by

$$\mathbb{Q}^n = \{x \in \mathbb{R}^n : 0 < x_r < 1\}, \quad \mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}. \tag{6.187}$$

As in (6.97) we decompose the boundary of the unit square in $\mathbb{R}^2$ as

$$\partial\mathbb{Q}^2 = \Gamma = \Gamma_1 \cup \Gamma_0, \quad \Gamma_1 \cap \Gamma_0 = \emptyset, \tag{6.188}$$

where Γ_0 collects the four corner points and Γ_1 the connecting line segments. Let temporarily $Q = \mathbb{Q}^2$ and let

$$d_0(x) = \mathrm{dist}(x, \Gamma_0), \quad x \in Q. \tag{6.189}$$

Then one has the sharp Hardy inequality

$$\int_Q \left| \frac{f(x)}{1 + |\log d_0(x)|} \right|^2 \frac{\mathrm{d}x}{d_0^2(x)} \le c \, \| f \, | H^1(Q) \|^2 \tag{6.190}$$

for the classical space $H^1(Q)$, whereas the restriction

$$H^1(Q, \Gamma_0) = \mathrm{re}_Q \; H^{1,\mathrm{rloc}}(\mathbb{R}^2 \setminus \Gamma_0) \tag{6.191}$$

of the refined localisation space $H^{1,\mathrm{rloc}}(\mathbb{R}^2 \setminus \Gamma_0)$ can be normed by

$$\| f \, | H^1(Q, \Gamma_0) \| = \Big(\sum_{|\alpha| \le 1} \| D^\alpha f \, | L_2(Q) \|^2 + \int_Q |f(x)|^2 \frac{\mathrm{d}x}{d_0(x)^2} \Big)^{1/2}. \tag{6.192}$$

Asking for u-Riesz bases as in Theorem 6.30 it turns out that the desired decoupling (6.117) for $H^1(Q) = F^1_{2,2}(Q)$ does not work. We discussed this point in Remark 5.50 with references to [Gri85], [Gri92]. On the other hand, according to Theorem 2.38 the space $H^{1,\mathrm{rloc}}(\mathbb{R}^2 \setminus \Gamma_0)$ has u-Riesz bases. By the specific construction one obtains that

$$\mathrm{tr}_{\Gamma_1} \; H^1(Q, \Gamma_0) = \widetilde{H}^{1/2}(\Gamma_1). \tag{6.193}$$

This results in the decomposition

$$H^1(Q, \Gamma_0) = \widetilde{H}^1(Q) \times \mathrm{Ext}_\Gamma \; \widetilde{H}^{1/2}(\Gamma_1) \tag{6.194}$$

as a counterpart of (6.117). Both factors have u-Riesz bases which can be clipped together as in Theorem 6.30. Then one obtains u-Riesz bases in $H^1(Q, \Gamma_0)$.

In other words, *if one reinforces* $H^1(Q)$ (satisfying the sharp Hardy inequality (6.190)) *by* $H^1(Q, \Gamma_0)$ *with* (6.192) *then one obtains the desired total decoupling of the boundary spaces and, as a consequence, u-Riesz bases.*

These considerations can be extended to n dimensions, $n \ge 3$. Let Γ_{n-2} be the $(n-2)$-dimensional faces of $\mathbb{Q}^n$ and let

$$d_{n-2}(x) = \mathrm{dist}(x, \Gamma_{n-2}), \quad x \in \mathbb{Q}^n, \tag{6.195}$$

be the counterpart of (6.189). Integrating (6.190) with respect to the remaining $n - 2$ dimensions one obtains the sharp Hardy inequality

$$\int_{\mathbb{Q}^n} \left| \frac{f(x)}{1 + |\log d_{n-2}(x)|} \right|^2 \frac{\mathrm{d}x}{d_{n-2}^2(x)} \le c \, \| f \, | H^1(\mathbb{Q}^n) \|^2 \tag{6.196}$$

whereas the corresponding reinforced spaces $H^1(\mathbb{Q}^n, \Gamma_{n-2})$ are normed by

$$\| f \, | H^1(\mathbb{Q}^n, \Gamma_{n-2}) \| = \Big(\sum_{|\alpha| \le 1} \| D^\alpha f \, | L_2(\mathbb{Q}^n) \|^2 + \int_{\mathbb{Q}^n} |f(x)|^2 \frac{\mathrm{d}x}{d_{n-2}^2(x)} \Big)^{1/2}. \tag{6.197}$$

Otherwise one is in the same position as in case of $n = 2$. One obtains the decomposition

$$H^1(\mathbb{Q}^n, \Gamma_{n-2}) = \widetilde{H}^1(\mathbb{Q}^n) \times \operatorname{Ext}_\Gamma \widetilde{H}^{1/2}(\Gamma_{n-1}) \tag{6.198}$$

and u-Riesz bases. This can be extended to other situations. One can replace the cube $\mathbb{Q}^n$ by the above unit ball $\mathbb{B}^n$ and the faces Γ_{n-2} by the equator

$$\mathbb{S}^{n-2} = \{x \in \mathbb{R}^n : |x| = 1, \ x_n = 0\}, \quad n \geq 3, \tag{6.199}$$

of the sphere $\partial\mathbb{B}^n$. The reinforced space $H^1(\mathbb{B}^n, \mathbb{S}^{n-2})$ is normed by (6.197) with $\mathbb{B}^n$ in place of $\mathbb{Q}^n$ and $d_{n-2}(x)$ as the distance of $x \in \mathbb{B}^n$ to $\mathbb{S}^{n-2}$. There is hardly any doubt that these sketchy comments can be applied to further spaces and other domains.

One takes sharp Hardy inequalities as they originate from the theory of envelopes as a guide and complements the (quasi)-norms of (classical) function spaces by reinforcing terms of the above type, paying special attention to the distance of (more or less) natural cutting faces (lines, edges). Employing the theory of the refined localisation spaces $F^{s,\mathrm{rloc}}_{pq}$ (wavelet bases) one tries to derive u-Riesz bases for these reinforced spaces.

6.3 Greedy bases

6.3.1 Definitions and basic assertions

First we recall some notation and well-known assertions. A set $\{e_l\}_{l=1}^\infty$ in a separable complex quasi-Banach space B is called a *basis* if any $b \in B$ can be uniquely represented as

$$b = \sum_{l=1}^\infty \lambda_l \, e_l, \quad \lambda_l \in \mathbb{C} \quad \text{(convergence in } B\text{)}. \tag{6.200}$$

Of course $\lambda_l(b)$ is linear in b. According to Proposition 6.35 any basis is a *Schauder basis*. This means that $\lambda_l(\cdot) \in B'$ are linear and continuous functionals. A basis $\{e_l\}_{l=1}^\infty$ is called an *unconditional basis* if for any rearrangement σ of $\mathbb{N}$ (one-to-one map of $\mathbb{N}$ onto itself) $\{e_{\sigma(l)}\}_{l=1}^\infty$ is again a basis,

$$b = \sum_{l=1}^\infty \lambda_{\sigma(l)} \, e_{\sigma(l)} \quad \text{(convergence in } B\text{)}, \tag{6.201}$$

for any $b \in B$ with (6.200). Standard bases of separable sequence spaces as considered in this book are always unconditional. One may think about the spaces $a^s_{pq}(\mathbb{Z}^\Omega)$ according to Definition 6.1 with $p < \infty, q < \infty$. But one should be aware that ℓ_2 and any other Banach space with a basis has also a basis which is not unconditional. This goes back to [PeS64] and may also be found in [AlK06], Section 9.5, pp. 235–40. We always assume that $\{e_l\}_{l=1}^\infty$ is *normalised*, hence $\|e_l|B\| \sim 1$, which means

$$0 < \inf_l \|e_l\,|B\| \le \sup_l \|e_l\,|B\| < \infty. \tag{6.202}$$

(Recall that $\{e_l\}_{l=1}^\infty$ is called *normed* if $\|e_l\,|B\| = 1$.) For a given basis $\{e_l\}_{l=1}^\infty$ let B_N be the collection of all $b \in B$ with at most $N \in \mathbb{N}$ non-vanishing coefficients $\lambda_l = \lambda_l(b)$ in (6.200). Then

$$s_N(b) = \inf\{\|b - b_N\,|B\| : \, b_N \in B_N\}, \quad N \in \mathbb{N}, \tag{6.203}$$

is called the best N-term approximation error (with respect to the given basis $\{e_l\}_{l=1}^\infty$). Let the coefficients $\lambda_l(b)$ in (6.200) be ordered by magnitude,

$$|\lambda_{l_1}(b)| \ge |\lambda_{l_2}(b)| \ge \cdots \ge |\lambda_{l_r}(b)| \ge \cdots . \tag{6.204}$$

Then

$$G_N(b) = \sum_{k=1}^N \lambda_{l_k}(b)\, e_{l_k}, \quad N \in \mathbb{N}, \tag{6.205}$$

is the *greedy algorithm*. One may ask of whether this distinguished (non-linear) approximation is comparable with $s_N(b)$ according to (6.203).

Definition 6.49. Let $\{e_l\}_{l=1}^\infty$ be a normalised basis in a separable complex quasi-Banach space.

(i) Then $\{e_l\}_{l=1}^\infty$ is said to be greedy if there is a constant C (≥ 1) such that for any $b \in B$ and any $N \in \mathbb{N}$,

$$\|b - G_N(b)\,|B\| \le C\, s_N(b). \tag{6.206}$$

(ii) Then $\{e_l\}_{l=1}^\infty$ is said to be democratic if there is a constant D (≥ 1) such that for any K tuplets

$$m_1 < m_2 < \cdots < m_K \quad \text{and} \quad l_1 < l_2 < \cdots < l_K \tag{6.207}$$

of natural numbers,

$$\Big\| \sum_{k=1}^K e_{m_k}\,|B \Big\| \le D \Big\| \sum_{k=1}^K e_{l_k}\,|B \Big\|. \tag{6.208}$$

Remark 6.50. Here we are interested only in the standard bases of complex quasi-Banach sequence spaces of type $a^s_{pq}(\mathbb{Z}^\Omega)$ and $a^s_{pq}(\mathbb{Z}_\Omega)$ with $a \in \{b, f\}$ according to Definitions 2.6 and 6.1 (or its $\mathbb{R}^n$-counterpart in Definition 1.3). This simplifies the situation. If $j \in \mathbb{N}_0$ has the same meaning as in Definitions 2.6, 6.1 then the normalised standard basis in these sequence spaces is given by

$$e_l = \big(0, \ldots, 0, 2^{-j(s-\frac{n}{p})}, 0, \ldots\big), \quad l \in \mathbb{N}, \tag{6.209}$$

appropriately numbered, $l = l(j, r)$.

Theorem 6.51. (i) *Let* $\{e_l\}_{l=1}^{\infty}$ *be a normalised basis in a* (*separable infinite-dimensional*) *complex Banach space. Then* $\{e_l\}_{l=1}^{\infty}$ *is greedy if, and only if, it is unconditional and democratic.*

(ii) *Let* $\{e_l\}_{l=1}^{\infty}$ *be a normalised unconditional basis in a* (*separable infinite-dimensional*) *complex quasi-Banach space. Then* $\{e_l\}_{l=1}^{\infty}$ *is greedy if, and only if, it is democratic.*

Proof. Part (i) for real Banach spaces is due to Konyagin and Temlyakov, [KoT99]. This proof may also be found in [AlK06], Theorem 9.6.3, pp. 242–44. The arguments given there apply also to complex Banach spaces. Presumably this can be extended to quasi-Banach spaces. But it is not immediately clear whether a greedy basis in a quasi-Banach space is also unconditional. This is the reason why we assumed in part (ii) that $\{e_l\}_{l=1}^{\infty}$ is unconditional. Otherwise the proof in [AlK06] does not use whether $\|\cdot\|$ is a norm or only a quasi-norm (in a complex quasi-Banach space). Then one obtains part (ii). □

Remark 6.52. At the first glance one is a little bit surprised that the positive word *democratic* is so closely connected with the somewhat negative word *greedy*. But this reflects simply human life as can be seen from the following quotation.

> Sir Winston Churchill (in the House of Commons, Nov. 11, 1947, perhaps reacting to corruption and other types of greediness in UK): *Indeed, it has been said that democracy is the worst form of government except all those other forms that have been tried from time to time.*

6.3.2 Greedy Riesz bases

The bases in spaces of type $A^s_{pq}(\Omega)$ on bounded Lipschitz domains Ω considered in the preceding sections are u-Riesz bases as introduced in Definition 6.5. Then one has the isomorphic map (6.19) onto the sequence spaces $a^s_{pq}(\mathbb{Z}^\Omega)$. But for the question of whether the standard basis of $a^s_{pq}(\mathbb{Z}^\Omega)$ is greedy or not it does not matter whether one deals with these sequence spaces or with the spaces $a^s_{pq}(\mathbb{Z}_\Omega)$, $a \in \{b, f\}$, according to Definition 2.6 incorporating now $\Omega = \mathbb{R}^n$. The standard basis (6.209) in $a^s_{pq}(\mathbb{Z}_\Omega)$ with $p < \infty$, $q < \infty$, is unconditional. Democracy and as a consequence of Theorem 6.51 also greediness of unconditional bases are invariant under isomorphic maps. Hence the question of whether the u-Riesz bases (u-wavelet bases) in diverse types of spaces B^s_{pq} and F^s_{pq} are greedy can be reduced to the problem of whether the standard basis (6.209) is democratic in the sequence spaces, say, $b^s_{pq}(\mathbb{Z}_\Omega)$ of $f^s_{pq}(\mathbb{Z}_\Omega)$ according to Definition 2.6. Recall that

$$b^s_{pp}(\mathbb{Z}_\Omega) = f^s_{pp}(\mathbb{Z}_\Omega), \quad 0 < p < \infty. \tag{6.210}$$

Proposition 6.53. *Let* $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q < \infty$.

(i) *The standard basis* (6.209) *is greedy in any* $f^s_{pq}(\mathbb{Z}_\Omega)$.

(ii) *The standard basis* (6.209) *is greedy in* $b^s_{pq}(\mathbb{Z}_\Omega)$ *if, and only if,* $p = q$.

Proof. Step 1. We prove (i). By Theorem 6.51 (ii) and the above comments it is sufficient to check that the unconditional basis (6.209) is democratic. Let $e_{l_1}, \dots, e_{l_K}$ be K elements (6.209) with $l_k = l_k(j_k, r_k)$ for e_{l_k}. For fixed $j \in \mathbb{N}_0$ we may assume that the characteristic functions χ_{jr} in (2.38) have pairwise disjoint supports. We used this possibility in this exposition several times with a reference to [T06], Section 1.5.3, pp. 18–19. Hence for fixed $x \in \Omega$ one obtains that

$$\sum_{k=1}^{K} |2^{-j_k(s-\frac{n}{p})}\, 2^{j_k s}\, \chi_{j_k,r_k}(x)|^q = \sum_{k=1}^{K} \delta_k\, 2^{j_k nq/p}, \quad x \in \Omega, \tag{6.211}$$

with $\delta_k = 0$ or $\delta_k = 1$. Let $j_{r_1} < j_{r_2} < \dots < j_{K'}$ be the j_ks contributing to (6.211) with $\delta_k = 1$. Then the right-hand side of (6.211) is equivalent to $2^{j_{K'} nq/p}$. Taking the $\frac{1}{q}$-power it follows that the integrand in (2.38) is equivalent to a function which is independent of q. One has

$$\begin{aligned} \Big\| \sum_{k=1}^{K} e_{l_k} \,|f^s_{pq}(\mathbb{Z}_\Omega)\Big\|^p &\sim \Big\| \sum_{k=1}^{K} e_{l_k} \,|f^s_{pp}(\mathbb{Z}_\Omega)\Big\|^p \\ &= \sum_{k=1}^{K} \int_\Omega 2^{j_k n} \big|\chi_{j_k,r_k}(x)\big|^p \,\mathrm{d}x \sim K. \end{aligned} \tag{6.212}$$

Hence, $\{e_l\}$ in (6.209) is a unconditional democratic basis in $f^s_{pq}(\mathbb{Z}_\Omega)$. Then it follows from Theorem 6.51 (ii) that the basis $\{e_l\}$ is greedy.

Step 2. By (6.210) and part (i) it follows that $\{e_l\}$ is greedy in $b^s_{pp}(\mathbb{Z}_\Omega)$. It remains to prove that $b^s_{pq}(\mathbb{Z}_\Omega)$ with $p \neq q$ is not greedy. First we choose e_{l_k} with $k = 1, \dots, K$ such that they have the same j in (6.209). Then one obtains from (2.37) that

$$\Big\| \sum_{k=1}^{K} e_{l_k} \,|b^s_{pq}(\mathbb{Z}_\Omega)\Big\| \sim K^{1/p}. \tag{6.213}$$

Choosing for e_{l_k} different levels of j, say $j_k = k$, then one obtains that

$$\Big\| \sum_{k=1}^{K} e_{l_k} \,|b^s_{pq}(\mathbb{Z}_\Omega)\Big\| \sim K^{1/q}. \tag{6.214}$$

This shows that the basis $\{e_l\}$ in (6.209) is not democratic and hence not greedy. □

All u-wavelet bases (= u-Riesz bases) constructed in this book in spaces of type A^s_{pq} with $A \in \{B, F\}$ generate isomorphic maps onto corresponding sequence spaces a^s_{pq} with $a \in \{b, f\}$. Then one can apply the above Proposition 6.53. Recall that $F^s_{pp} = B^s_{pp}$ and similarly $f^s_{pp} = b^s_{pp}$. Saying *if, and only if,* $A = F$, includes B^s_{pp}, but excludes B^s_{pq} with $p \neq q$.

Theorem 6.54. (i) *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q < \infty$. Then the wavelet bases for $A^s_{pq}(\mathbb{R}^n)$ according to Theorem 1.20 are greedy if, and only if, $A = F$.*

(ii) *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. The wavelet bases for $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ with $0 < p < \infty$, $0 < q < \infty$, $s > \sigma_{pq}$, according to Theorem 2.38 are greedy.*

(iii) *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. The wavelet bases for $L_p(\Omega)$ with $1 < p < \infty$ according to Theorem 2.36 are greedy.*

(iv) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition 3.4* (iii) *or a bounded interval on $\mathbb{R}$ if $n = 1$. Let $\bar{A}^s_{pq}(\Omega)$ be the spaces in* (3.45), (3.46) *with $p < \infty$, $q < \infty$. Then the wavelet bases for $\bar{A}^s_{pq}(\Omega)$ according to Theorem 3.23 are greedy if, and only if, $A = F$.*

(v) *Let Ω be either a bounded interval on $\mathbb{R}$ ($n = 1$) or a bounded planar C^∞ domain in $\mathbb{R}^2$ ($n = 2$). Then the u-wavelet bases for the spaces $A^s_{pq}(\Omega)$ according to Theorem 5.35 are greedy if, and only if, $A = F$.*

(vi) *Let Ω be the special bounded C^∞ domain in $\mathbb{R}^n$ with $n \geq 3$ considered in Theorem 5.38. Then the corresponding u-wavelet bases for the spaces $A^s_{pq}(\Omega)$ covered are greedy if, and only if, $A = F$.*

(vii) *Let Ω be a cellular domain in $\mathbb{R}^n$ with $n \geq 2$ according to Definition 6.9 (which includes cubes and bounded C^∞ domains). The u-Riesz bases for the spaces $A^s_{pq}(\Omega)$ covered by Theorem 6.32 are greedy if, and only if, $A = F$.*

Proof. In all cases one has isomorphic maps of the indicated u-Riesz bases, u-wavelet bases, onto spaces of type $a^s_{pq}(\mathbb{Z}_\Omega)$ and their appropriate modifications. Application of Proposition 6.53 gives the desired result. □

Remark 6.55. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and let $\{\Phi^j_r\}$ be the orthonormal u-wavelet basis in $L_2(\Omega)$ according to Theorem 2.36 where any smoothness $u \in \mathbb{N}$ is admitted. By this theorem and part (iii) of the above theorem, $\{\Phi^j_r\}$ is also a greedy basis in $L_p(\Omega)$ where $1 < p < \infty$. At least for L_p-spaces it is reasonable to ask for greedy bases of smoothness $u = 0$. In Section 2.5.1 we recalled in (2.163) the classical Haar system

$$H = \{H^j_{G,m} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\} \tag{6.215}$$

in $\mathbb{R}^n$ and introduced in (2.183) its Ω-adapted counterpart H^Ω.

Theorem 6.56. *Let $1 < p < \infty$ and $n \in \mathbb{N}$.*

(i) *The classical Haar system in (6.215) is a greedy basis in $L_p(\mathbb{R}^n)$.*

(ii) *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Then the Haar system H^Ω according to (2.183) is a greedy basis in $L_p(\Omega)$.*

Proof. This follows immediately from the famous Paley–Littlewood assertion in Theorem 2.40 (i) for the spaces $L_p(\mathbb{R}^n)$, its Ω-counterpart in Theorem 2.44 and Proposition 6.53 (adapted to the above situation). □

Remark 6.57. If one has a basis in a space of type A^s_{pq} which is isomorphic to the standard basis of a related sequence space a^s_{pq} then one obtains by Proposition 6.53 a greedy basis if, and only if, $a = f$. Theorems 6.54 and 6.56 are examples. But this applies also to the weighted spaces $A^s_{pq}(\mathbb{R}^n, w)$ according to Theorem 1.26 and the periodic spaces $A^s_{pq}(\mathbb{T}^n)$ in Theorem 1.37. One can extend these assertions to anisotropic spaces $A^{s,\alpha}_{pq}(\mathbb{R}^n)$, where corresponding wavelet expansions may be found in [T06], Section 5.2.2, pp. 252–54.

Remark 6.58. Greedy bases came up at the end of the 1990s by the work of Temlyakov, where [KoT99] is a nowadays often quoted paper. We refer to [AlK06], Section 9.6, and the survey [Woj03] for further information and references. The considerations are mostly restricted to real Banach spaces. But it is also mentioned that the assertions can be extended to complex Banach spaces and, more generally, to complex quasi-Banach spaces, as we did in Theorem 6.51. Here we are mainly interested in the standard basis (6.209) in spaces of type $a^s_{pq}(\mathbb{Z}_\Omega)$. Then everything can be done directly. But assertions of this type for sequence spaces of f-type and b-type are not new. We refer in this context also to [GaH04], Theorem 2.1, dealing with anisotropic (not necessarily diagonal) sequence spaces. The first observation about the greediness of Haar bases in L_p-spaces with $1 < p < \infty$ in $\mathbb{R}^n$ and on cubes goes back to [Tem98]. The search for greedy bases in L_1, Lorentz spaces $L_{p,q}$, Orlicz spaces L^Φ and other rearrangement invariant spaces seems to be a somewhat tricky business. It is subject to several papers. We mention [GHM07], [Woj06] and the references within. Greedy bases in so-called α-modulation spaces with brushlets in place of wavelets have been studied in [BoN06a], [BoN06b].

6.4 Dichotomy: traces versus density

6.4.1 Preliminaries

Traces of spaces $A^s_{pq}(\mathbb{R}^n)$ with $2 \le n \in \mathbb{N}$, $s > 0$ and $1 \le p, q \le \infty$ on hyper-planes $\mathbb{R}^d$ with $n > d \in \mathbb{N}$ or on boundaries $\Gamma = \partial\Omega$ of smooth domains Ω have been considered since a long time. They played also a role in this exposition. One may consult Sections 5.1.1, 5.1.3 and also Section 6.1. There is always the question about the alternative (dichotomy)

- either $D(\mathbb{R}^n \setminus \Gamma)$ is dense in $A^s_{pq}(\mathbb{R}^n)$,
- or $A^s_{pq}(\mathbb{R}^n)$ has a trace on Γ as in (5.5).

One has to say what is meant by traces. If $\Gamma = \partial\Omega$ is the boundary of a bounded C^∞ domain furnished naturally with the $(n-1)$-dimensional Hausdorff measure $\mu = \mathcal{H}^{n-1}_\Gamma$ then the question (5.5) is reasonable, especially if s, p, q are restricted as in (5.3). However if $\mu = \mathcal{H}^{n-1}_\Gamma + \delta$, where δ is the Dirac measure at $0 \in \Gamma$, then the situation is different. For pointwise traces of $A^s_{pq}(\mathbb{R}^n)$ one needs $s \geq n/p$, whereas the density requires $s \leq 1/p$. There is a gap and the dichotomy we are asking for cannot be expected. This suggests to restrict the above question to isotropic Radon measures.

We assume that the reader is familiar with basic measure and integration theory. Short descriptions of what is needed may be found in [Mat95], pp. 7–13, [Fal85], pp. 1–6, [T97], pp. 1–2, or [T06], Section 1.12.2, pp. 80–81. With exception of the Lebesgue measure in $\mathbb{R}^d$, considered as a hyper-plane in $\mathbb{R}^n$, we always assume that μ is a Radon measure in $\mathbb{R}^n$ with

$$0 < \mu(\mathbb{R}^n) < \infty \quad \text{and} \quad \operatorname{supp} \mu = \Gamma \text{ compact.} \tag{6.216}$$

Recall that there is an one-to-one relation between these Radon measures μ and the tempered distribution T_μ,

$$T_\mu : \varphi \mapsto \int_{\mathbb{R}^n} \varphi(x)\, \mu(\mathrm{d}x), \quad \varphi \in S(\mathbb{R}^n), \tag{6.217}$$

generated by μ. This justifies to identify μ with T_μ and to write $\mu \in S'(\mathbb{R}^n)$. It is essentially the famous Riesz representation of continuous functionals on $C_0(\mathbb{R}^n)$ (the collection of all continuous functions on $\mathbb{R}^n$ tending to zero at infinity). Details may be found in [T06], Proposition 1.123, pp. 80–81. There is an interesting recent extension to unbounded tempered Radon measures, [Kab08]. But we stick here at (6.216). Let $L_r(\Gamma,\mu)$ with $0 < r < \infty$ be the usual complex quasi-Banach spaces, quasi-normed by

$$\|g \,|L_r(\Gamma,\mu)\| = \left(\int_{\mathbb{R}^n} |g(\gamma)|^r \,\mu(\mathrm{d}\gamma) \right)^{1/r} = \left(\int_{\Gamma} |g(\gamma)|^r \,\mu(\mathrm{d}\gamma) \right)^{1/r}. \tag{6.218}$$

If $r \geq 1$ then $g \in L_r(\Gamma,\mu)$ generates as in (6.217) with the complex Radon measure $g\mu$ in place of μ a tempered distribution and one has again an one-to-one relation between

$$g \in L_r(\Gamma,\mu) \quad \text{and} \quad g\mu \in S'(\mathbb{R}^n). \tag{6.219}$$

However if $0 < r < 1$ then this is no longer the case. Just this point is responsible for some curious effects which will be discussed later on. By the above comments it is reasonable to assume that the measure μ with (6.216) is isotropic. As before a ball in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ and of radius ϱ is denoted by $B(x,\varrho)$. A non-negative function h on the unit interval $[0,1]$ is called strictly increasing if $h(t_1) > h(t_2)$ for $t_1 > t_2$. Then a Radon measure μ with (6.216) is called *isotropic* if there is a continuous strictly increasing function h on the interval $[0,1]$ with $h(0) = 0$ and

$$\mu\big(B(\gamma,\varrho)\big) \sim h(\varrho) \quad \text{where } \gamma \in \Gamma \text{ and } 0 < \varrho < 1, \tag{6.220}$$

(the equivalence constants are independent of γ and ϱ). We refer to [T06], Section 1.15.1, pp. 95–97, for details and, in particular, for a criterion which functions h generate a measure μ with (6.220). It goes back to [Bri03], Theorem 2.7. One may also consult [Bri04]. It might be of interest to study dichotomy in the context of measures μ with (6.216), (6.220). But we restrict ourselves to the most distinguished isotropic measures. These are the so-called d-sets Γ as supports of measures μ with (6.216) and

$$\mu\big(B(\gamma,\varrho)\big) \sim \varrho^d, \quad \gamma \in \Gamma,\ 0 < \varrho < 1,\ 0 < d < n. \tag{6.221}$$

It is well known that for given Γ and given d a measure μ with (6.216), (6.221) is equivalent to the restriction $\mathcal{H}^d_\Gamma$ of the Hausdorff measure $\mathcal{H}^d$ in $\mathbb{R}^n$ to Γ. A short proof may be found in [T97], Theorem 3.4, pp. 5–6. This justifies to speak simply about d-sets Γ and to simplify in this case $L_r(\Gamma,\mu)$ by $L_r(\Gamma)$ and (6.218) by

$$\|g\,|L_r(\Gamma)\| = \bigg(\int_\Gamma |g(\gamma)|^r\,\mu(\mathrm{d}\gamma)\bigg)^{1/r}, \quad \mu \sim \mathcal{H}^d_\Gamma. \tag{6.222}$$

This coincides also with our previous notation in (5.4), (5.18). If $L_r(\Gamma)$ or, more general, $L_r(\Gamma,\mu)$ with (6.216), (6.218) is chosen as a target space for some $f \in A^s_{pq}(\mathbb{R}^n)$ one may ask to which extent the trace (if exists) depends on the chosen source and target spaces. Roughly speaking, it should depend only on f. This is also largely the case and will be justified below. But for this purpose it is desirable to have a constructive procedure for the well-known assertion that $D(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ are dense in $A^s_{pq}(\mathbb{R}^n)$ with $p<\infty$, $q<\infty$. The spaces $A^s_{pq}(\mathbb{R}^n)$ have the same meaning as in (1.95) and Definition 1.1.

Proposition 6.59. *There are linear operators*

$$I_{j,l}:\quad S'(\mathbb{R}^n) \hookrightarrow S(\mathbb{R}^n), \quad j\in\mathbb{N},\ l\in\mathbb{N}, \tag{6.223}$$

such that for all $f\in A^s_{pq}(\mathbb{R}^n)$ with $s\in\mathbb{R}$, $0<p<\infty$, $0<q<\infty$, and $\varepsilon>0$,

$$\|f - I_{j,l}f\,|A^s_{pq}(\mathbb{R}^n)\| \le \varepsilon \quad \text{if } j \ge j_\varepsilon(f, A^s_{pq}),\ l \ge l_\varepsilon(f,j,A^s_{pq}). \tag{6.224}$$

Proof. Let φ_0 be the same standard function as in (1.5) and let $f \in A^s_{pq}(\mathbb{R}^n)$ with $p<\infty$, $q<\infty$. Then one has

$$f_j = \big(\varphi_0(2^{-j}\cdot)\widehat{f}\,\big)^\vee \to f \quad \text{in } A^s_{pq}(\mathbb{R}^n) \text{ if } j\to\infty \tag{6.225}$$

as a consequence of Definition 1.1, Fourier multiplier assertions and, in case of F-spaces, Lebesgue's dominated convergence theorem, [Mall95], p. 37. Recall that f_j is an entire analytic function. Let $\psi\in S(\mathbb{R}^n)$ with $\psi(0)=1$ such that $\widehat{\psi}$ has compact support. Then

$$f^l_j = \psi(2^{-l}\cdot)f_j \in S(\mathbb{R}^n) \quad \text{and} \quad \operatorname{supp}\widehat{f^l_j} \subset \big\{y\in\mathbb{R}^n: |y|\le c\,2^j\big\} \tag{6.226}$$

for some $c>0$, all $j\in\mathbb{N}$ and all $l\in\mathbb{N}$. Here $f^l_j\in S(\mathbb{R}^n)$ follows from the Paley–Wiener–Schwartz theorem, [T83], p. 13, whereas the second assertion in (6.226) is a consequence of the convolution

$$\widehat{f^l_j} = 2^{nl}\,\widehat{\psi}(2^l\cdot)\star\widehat{f_j}.$$

One obtains again by Lebesgue's dominated convergence theorem that

$$\| f_j^l - f_j \,|A^s_{pq}(\mathbb{R}^n)\| \le c_j \,\|\big(1-\psi(2^{-l}\,\cdot)\big)\, f_j \,|L_p(\mathbb{R}^n)\| \to 0 \tag{6.227}$$

and hence

$$f_j^l \to f_j \ \text{ in } \ A^s_{pq}(\mathbb{R}^n) \ \text{ if } l \to \infty. \tag{6.228}$$

With $I_{j,l} f = f_j^l$ one obtains (6.224). □

Remark 6.60. It is known since a long time that $S(\mathbb{R}^n)$ and also $D(\mathbb{R}^n)$ are dense in $A^s_{pq}(\mathbb{R}^n)$ with $p < \infty$, $q < \infty$, [T83], Theorem, p. 48. But it will be helpful for us to have the explicit universal approximating sequence f_j^l of f.

6.4.2 Traces

Although we are mainly interested in d-sets it is reasonable to deal first with Radon measures μ in $\mathbb{R}^n$ according to (6.216) and the related spaces $L_r(\Gamma,\mu)$ quasi-normed by (6.218). Let $A^s_{pq}(\mathbb{R}^n)$ with $A \in \{B, F\}$ be the spaces introduced in Definition 1.1.

Definition 6.61. Let μ be a Radon measure in $\mathbb{R}^n$ with (6.216). Let $0 < r < \infty$,

$$0 < p < \infty, \quad 0 < q < \infty, \quad s \in \mathbb{R}. \tag{6.229}$$

Let for some $c > 0$,

$$\|\varphi \,|L_r(\Gamma,\mu)\| \le c\,\|\varphi \,|A^s_{pq}(\mathbb{R}^n)\| \quad \text{for all } \varphi \in S(\mathbb{R}^n). \tag{6.230}$$

Then the trace operator tr_μ,

$$\mathrm{tr}_\mu :\ A^s_{pq}(\mathbb{R}^n) \hookrightarrow L_r(\Gamma,\mu), \tag{6.231}$$

is the completion of the pointwise trace $(\mathrm{tr}_\mu\,\varphi)(\gamma) = \varphi(\gamma)$ with $\varphi \in S(\mathbb{R}^n)$ and $\gamma \in \Gamma$.

Remark 6.62. Since $S(\mathbb{R}^n)$ is dense in $A^s_{pq}(\mathbb{R}^n)$ it follows by standard arguments that this definition makes sense and that the outcome

$$\mathrm{tr}_\mu\, f \in L_r(\Gamma,\mu), \quad f \in A^s_{pq}(\mathbb{R}^n), \tag{6.232}$$

is independent of the chosen approximating sequence

$$\varphi_j \to f \ \text{ in } \ A^s_{pq}(\mathbb{R}^n), \quad \varphi_j \in S(\mathbb{R}^n), \tag{6.233}$$

on the source side. On the target side one has

$$\mathrm{tr}_\mu\, \varphi_j \to \mathrm{tr}_\mu\, f \ \text{ in } \ L_r(\Gamma,\mu). \tag{6.234}$$

By standard arguments of measure theory, see e.g. [Mall95], Theorem 5.2.7, p. 23, Lemma 9.3.3, p. 51, it follows that there is a subsequence of $\{\varphi_j\}$ which converges

μ-a.e. to $\operatorname{tr}_\mu f$. Identifying this subsequence with $\{\varphi_j\}$ we may assume in addition to (6.234) that

$$\varphi_j(\gamma) = (\operatorname{tr}_\mu \varphi_j)(\gamma) \to (\operatorname{tr}_\mu f)(\gamma) \quad \mu\text{-a.e. on } \Gamma. \tag{6.235}$$

One obtains that $\operatorname{tr}_\mu f$ is μ-a.e. independent of r as long as one has (6.230). This applies in particular to $0 < \tilde{r} \le r$ with (6.230) as a consequence of $\mu(\Gamma) < \infty$ and Hölder's inequality. There is a similar situation on the source side assuming that one has (6.230) for $A^{s_1}_{p_1 q_1}(\mathbb{R}^n)$ and $A^{s_2}_{p_2 q_2}(\mathbb{R}^n)$ instead of $A^s_{pq}(\mathbb{R}^n)$ on the right-hand side and that

$$f \in A^{s_1}_{p_1 q_1}(\mathbb{R}^n) \cap A^{s_2}_{p_2 q_2}(\mathbb{R}^n). \tag{6.236}$$

Then one can rely on the distinguished approximation $I_{j,l} f$ of f according to Proposition 6.59 where one may assume that it applies to both spaces in (6.236). Hence one has a common approximating sequence, and $\operatorname{tr}_\mu f$ is μ-a.e. the same for both spaces in (6.236). If (6.230) holds both for $L_{r_1}(\Gamma_1, \mu_1)$ and $L_{r_2}(\Gamma_2, \mu_2)$ then it is also valid for $L_r(\Gamma, \mu)$ with $\mu = \mu_1 + \mu_2$ and $r = \min(r_1, r_2)$. By the above considerations, $\operatorname{tr}_{\mu_1}$ and $\operatorname{tr}_{\mu_2}$ are restrictions of tr_μ. In other words,

> *for individual elements f the traces are independent of the source spaces and of the target spaces as long as one has* (6.230) *and whenever comparison makes sense.*

In (5.6) we dealt with the more specific trace spaces $L_r(\Gamma)$, $r \ge 1$. Then one has (5.8). But this requires the necessary restriction

$$A^s_{pq}(\mathbb{R}^n) \subset L_1^{\mathrm{loc}}(\mathbb{R}^n), \quad \text{hence } s \ge \sigma_p = n\Big(\frac{1}{p} - 1\Big)_+, \tag{6.237}$$

which does not fit in our concept. As will be seen, the above definitions apply even to some singular distributions, for example the δ-distribution, for which (5.8) does not make any sense.

At the beginning of Section 6.4.1 we have given a first rough description of what is meant by dichotomy. Now it is clear that traces must be understood according to Definition 6.61. One has the following almost obvious observation. If Γ is a compact set in $\mathbb{R}^n$ or the hyper-plane $\mathbb{R}^d$ in $\mathbb{R}^n$ then we abbreviate now

$$D_\Gamma = D(\mathbb{R}^n \setminus \Gamma). \tag{6.238}$$

Proposition 6.63. *Let $A^s_{pq}(\mathbb{R}^n)$ and $L_r(\Gamma, \mu)$ be as in Definition 6.61. If D_Γ is dense in $A^s_{pq}(\mathbb{R}^n)$ then there is no $c > 0$ with* (6.230).

Proof. We assume that there is a constant $c > 0$ with (6.230). We approximate a function φ which is identically 1 near Γ by D_Γ-functions. Then one has by Remark 6.62 that $\operatorname{tr}_\mu \varphi = 0$ μ-a.e. This contradicts $\mu(\Gamma) > 0$. □

As said at the beginning of Section 6.4.1 it does not make much sense to deal with the dichotomy problem in the context of general Radon measures μ according to

(6.216). We concentrate on d-sets Γ with (6.221) where $\mu = \mathcal{H}^d_\Gamma$ is the restriction of the Hausdorff measure $\mathcal{H}^d$ in $\mathbb{R}^n$ to Γ. Then we simplify $L_r(\Gamma,\mu)$ by $L_r(\Gamma)$ with (6.222). Similarly we put

$$\mathrm{tr}_\Gamma = \mathrm{tr}_\mu, \quad \mu \sim \mathcal{H}^d_\Gamma, \quad \Gamma = \mathrm{supp}\,\mu, \tag{6.239}$$

in this case, in agreement with our previous notation in (5.5), (5.6). As far as traces on d-sets are concerned one has the following assertion.

Proposition 6.64. *Let Γ be a compact d-set in $\mathbb{R}^n$ with $0 < d < n$ furnished with the Radon measure $\mu = \mathcal{H}^d_\Gamma$ satisfying* (6.221). *Then*

$$\mathrm{tr}_\Gamma\, B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\Gamma) \quad \text{if } 0 < p < \infty,\ 0 < q \le \min(1,p), \tag{6.240}$$

and

$$\mathrm{tr}_\Gamma\, F^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\Gamma) \quad \text{if } 0 < p \le 1,\ 0 < q < \infty. \tag{6.241}$$

Remark 6.65. Of course, (6.240), (6.241) means that the linear and bounded trace operator $\mathrm{tr}_\Gamma = \mathrm{tr}_\mu$ according to Definition 6.61 exists and that it is a map onto $L_p(\Gamma) = L_p(\Gamma,\mu)$. These assertions have a substantial history. First we remark that Γ can be replaced by $\mathbb{R}^d$ with $n > d \in \mathbb{N}$ and the d-dimensional Lebesgue measure, interpreted as d-dimensional hyper-plane in $\mathbb{R}^n$. Then one has in obvious notation,

$$\mathrm{tr}\ B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\mathbb{R}^d) \quad \text{if } 0 < p < \infty,\ 0 < q \le \min(1,p), \tag{6.242}$$

and

$$\mathrm{tr}\ F^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\mathbb{R}^d) \quad \text{if } 0 < p \le 1,\ 0 < q < \infty. \tag{6.243}$$

The first proof of (6.242) for $1 \le p < \infty$ (and $d = n-1$) goes back to [Pee75], [Gol79], [BuG79]. This has been extended to $0 < p < \infty$ in [FrJ85]. The F-counterpart, hence (6.243) (with $d = n-1$), is due to [Tri89], [FrJ90]. Further details may be found in [T92], Section 4.4.3, pp. 220–21. The step from $\mathbb{R}^d$ to d-sets requires new technical instruments, especially atomic decompositions. Proposition 6.64 coincides with [T06], Proposition 1.172. But it goes back to [T97], Corollary 18.12, p. 142, and the later observation that any d-set with $d < n$ is porous (called ball condition in [T97]) according to Definition 3.4 (i). We refer to [T01], Remark 9.19, pp. 140–41, and also to the above Section 3.2.4. However we wish to mention that one finds some assertions of this type in an earlier paper by Yu. V. Netrusov, [Net90], Corollary, p. 193.

6.4.3 Dichotomy

The above discussions suggest that the dichotomy *traces versus density* requires isotropic measures μ with (6.220). We deal here exclusively with d-sets and the related target spaces $L_r(\Gamma)$ according to (6.221), (6.222). The spaces $A^s_{pq}(\mathbb{R}^n)$ have the same meaning as above, (1.95) and Definition 1.1. Let D_Γ and tr_Γ be as in (6.238) and (6.239).

Definition 6.66. Let $n \in \mathbb{N}$ and $0 < p < \infty$. Let

$$A_p(\mathbb{R}^n) = \{A^s_{pq}(\mathbb{R}^n) : 0 < q < \infty, \ s \in \mathbb{R}\}. \tag{6.244}$$

Let $0 < d < n$ and let $\Gamma = \operatorname{supp} \mu$ be a compact d-set in $\mathbb{R}^n$ with $\mu = \mathcal{H}^d_\Gamma$. Let $\sigma \in \mathbb{R}$. Then

$$\mathbb{D}\big(A_p(\mathbb{R}^n), L_p(\Gamma)\big) = (\sigma, u) \quad \text{with } 0 < u < \infty, \tag{6.245}$$

is called the dichotomy of $\{A_p(\mathbb{R}^n), L_p(\Gamma)\}$ if

$$\operatorname{tr}_\Gamma : A^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma) \text{ exists for } \begin{cases} s > \sigma, & 0 < q < \infty, \\ s = \sigma, & 0 < q \le u, \end{cases} \tag{6.246}$$

and

$$D_\Gamma \text{ is dense in } A^s_{pq}(\mathbb{R}^n) \text{ for } \begin{cases} s = \sigma, & u < q < \infty, \\ s < \sigma, & 0 < q < \infty. \end{cases} \tag{6.247}$$

Furthermore,

$$\mathbb{D}\big(A_p(\mathbb{R}^n), L_p(\Gamma)\big) = (\sigma, 0) \tag{6.248}$$

means that

$$\left.\begin{array}{l} \operatorname{tr}_\Gamma \ \text{ exists for } s > \sigma, 0 < q < \infty, \\ D_\Gamma \text{ is dense in } A^s_{pq}(\mathbb{R}^n) \text{ for } s \le \sigma, 0 < q < \infty, \end{array}\right\} \tag{6.249}$$

and

$$\mathbb{D}\big(A_p(\mathbb{R}^n), L_p(\Gamma)\big) = (\sigma, \infty) \tag{6.250}$$

means that

$$\left.\begin{array}{l} \operatorname{tr}_\Gamma \ \text{ exists for } s \ge \sigma, 0 < q < \infty, \\ D_\Gamma \text{ is dense in } A^s_{pq}(\mathbb{R}^n) \text{ for } s < \sigma, 0 < q < \infty. \end{array}\right\} \tag{6.251}$$

Remark 6.67. Recall that $D(\mathbb{R}^n)$ is dense in $A^s_{pq}(\mathbb{R}^n)$ with $p < \infty, q < \infty$. Furthermore for fixed $A = B$ or $A = F$ and $0 < p < \infty$ one has the continuous embeddings

$$A^{s_1}_{pq_1}(\mathbb{R}^n) \hookrightarrow A^{s_2}_{pq_2}(\mathbb{R}^n) \quad \text{for } s_2 < s_1 \text{ and } 0 < q_1, q_2 < \infty \tag{6.252}$$

and

$$A^{s}_{pq_1}(\mathbb{R}^n) \hookrightarrow A^{s}_{pq_2}(\mathbb{R}^n) \quad \text{for } s \in \mathbb{R} \text{ and } 0 < q_1 \le q_2 < \infty. \tag{6.253}$$

Together with Proposition 6.63 it follows that the above definition makes sense. If one deals with more general isotropic measures (6.220) and with more general source spaces of B-type and F-type then it might be reasonable to replace $q \le u$ in (6.246) by $q < u$ and to leave open (at least in the definition) what happens at the breaking point $q = u$. But in our case one has the additional information that the sharp breaking point (σ, u) is on the trace side.

Theorem 6.68. *Let $n \in \mathbb{N}$, $0 < d < n$ and $0 < p < \infty$. Let Γ be a compact d-set in $\mathbb{R}^n$ and $\mu = \mathcal{H}^d_\Gamma$. Then*

$$\mathbb{D}\big(B_p(\mathbb{R}^n), L_p(\Gamma)\big) = \begin{cases} \big(\frac{n-d}{p}, 1\big) & \text{if } p > 1, \\ \big(\frac{n-d}{p}, p\big) & \text{if } p \le 1, \end{cases} \tag{6.254}$$

and

$$\mathbb{D}\big(F_p(\mathbb{R}^n), L_p(\Gamma)\big) = \begin{cases} \big(\frac{n-d}{p}, 0\big) & \text{if } p > 1, \\ \big(\frac{n-d}{p}, \infty\big) & \text{if } p \le 1. \end{cases} \tag{6.255}$$

Proof. Step 1. In the Steps 3 and 5 below we prove that

$$D_\Gamma \text{ is dense in } B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) \ \text{ if } \ 0 < p < \infty,\ q > \min(1,p). \tag{6.256}$$

Then (6.254) follows from (6.240) and the comments in Remark 6.67. Since $D(\mathbb{R}^n)$ is dense in $F^s_{pq}(\mathbb{R}^n)$, $q < \infty$, one obtains by (6.256) that

$$D_\Gamma \text{ is dense in } F^s_{pq}(\mathbb{R}^n), \quad s < \frac{n-d}{p},\ 0 < q < \infty. \tag{6.257}$$

Then (6.255) with $p \le 1$ is a consequence of (6.241). By (6.240) it follows that tr_Γ exists for all spaces $F^s_{pq}(\mathbb{R}^n)$ with $s > \frac{n-d}{p}$. Hence it remains to prove that

$$D_\Gamma \text{ is dense in } F^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) \ \text{ if } \ 1 < p < \infty,\ 0 < q < \infty. \tag{6.258}$$

In other words, the theorem follows from (6.256) (subject to Steps 3 and 5) and (6.258) (subject to Step 4).

Step 2. We begin with a preparation. We wish to construct a sequence $\{\varphi^J\}_{J=1}^\infty \subset D(\mathbb{R}^n)$ with

$$\varphi^J(x) = 1 \quad \text{in an open neighbourhood of } \Gamma \tag{6.259}$$

(depending on J) and

$$\varphi^J \to 0 \ \text{ in } \ B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) \quad \text{if } J \to \infty,\ p \ge 1,\ q > 1. \tag{6.260}$$

For given $j \in \mathbb{N}$ we cover a neighbourhood of Γ with balls $B_{j,m}$ centred at Γ and of radius 2^{-j}, where $m = 1, \dots, M_j$ with $M_j \sim 2^{jd}$ such that there is a resolution of unity,

$$\sum_{m=1}^{M_j} \varphi_{j,m}(x) = 1 \ \text{ near } \ \Gamma, \quad 0 \le \varphi_{j,m} \in D(B_{j,m}), \tag{6.261}$$

with the usual properties,

$$|D^\gamma \varphi_{j,m}(x)| \le c_\gamma\, 2^{j|\gamma|}, \quad \gamma \in \mathbb{N}^n_0. \tag{6.262}$$

For $2 \le J \in \mathbb{N}$ let $J' \in \mathbb{N}$ be such that

$$\sum_{j=J}^{J'+1} r_j = 1 \quad \text{with } r_j = j^{-1} \text{ if } J \le j \le J',\ 0 < r_{J'+1} \le (J'+1)^{-1}. \tag{6.263}$$

Then

$$\varphi^J(x) = \sum_{j=J}^{J'+1} r_j\, 2^{-j\frac{d}{p}} \sum_{m=1}^{M_j} 2^{\frac{jd}{p}}\, \varphi_{j,m}(x), \quad x \in \mathbb{R}^n, \tag{6.264}$$

is an atomic decomposition in $B^s_{pq}(\mathbb{R}^n)$ according to Theorem 1.7 and Definition 1.5 for $s = \frac{n-d}{p}$, $p \ge 1$, $q > 1$, with $L = 0$ (no moment conditions). We used $s - \frac{n}{p} = -\frac{d}{p}$. One obtains by Theorem 1.7 that

$$\left\| \varphi^J \,|B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\right\|^q \le c \sum_{j=J}^{J'+1} r^q_j\, 2^{-j\frac{dq}{p}} \Big(\sum_{m=1}^{M_j} 1 \Big)^{q/p} \le c' \sum_{j=J}^{\infty} j^{-q} \sim J^{1-q}. \tag{6.265}$$

This proves (6.259), (6.260).

Step 3. We prove (6.256) for $p > 1$, $q > 1$. It is sufficient to approximate $f \in D(\mathbb{R}^n)$ in $B^s_{pq}(\mathbb{R}^n)$ with $s = \frac{n-d}{p}$ by functions $f^J \in D_\Gamma$. Let φ^J be the functions according to (6.259), (6.260) and let

$$f = f_J + f^J \quad \text{with } f_J = \varphi^J f \text{ and } f^J = (1-\varphi^J) f \in D_\Gamma. \tag{6.266}$$

By the pointwise multiplier theorem in [T92], Corollary, p. 205, one has for $\varrho > \frac{n-d}{p}$, some $c > 0$, all $f \in D(\mathbb{R}^n)$ and all φ^J that

$$\| f_J \,|B^s_{pq}(\mathbb{R}^n)\| \le c\, \| f \,|\mathcal{C}^\varrho(\mathbb{R}^n)\| \cdot \|\varphi^J \,|B^s_{pq}(\mathbb{R}^n)\| \to 0 \tag{6.267}$$

if $J \to \infty$ where we used (6.260). This proves (6.256) for $p > 1$, $q > 1$.

Step 4. We prove (6.258). As mentioned at the end of Remark 6.65 the compact d-set Γ with $d < n$ is porous. Then one can apply [T06], (9.90), Definition 9.18, pp. 392–93, to the atomic decomposition (6.264) in $F^s_{pq}(\mathbb{R}^n)$ with $s = \frac{n-d}{p}$, $p > 1$, $q \ge 1$ (where one may choose $L = 0$ in Theorem 1.7, no moment conditions) with the outcome

$$\lim_{J\to\infty} \| \varphi^J \,|F^s_{pq}(\mathbb{R}^n)\| = \lim_{J\to\infty} \|\varphi^J \,|B^s_{pp}(\mathbb{R}^n)\| = 0. \tag{6.268}$$

For this type of argument one may also consult [T01], pp. 142–43. If $0 < q < 1$ then it may happen that (6.264) is no longer an atomic decomposition in $F^s_{pq}(\mathbb{R}^n)$ according to Theorem 1.7 since one may need moment conditions. Because Γ is porous one can complement $\varphi_{j,m}$ outside of Γ in an appropriate way such that one has the needed moment conditions. Details may be found in [T97], p. 143, with a reference to [TrW96]. After this modification one has (6.268) for all $s = \frac{n-d}{p}$, $p > 1$, $0 < q < \infty$. Then one obtains (6.258) in the same way as in Step 3.

Step 5. We prove (6.256) for $p < q$. This covers in particular the remaining cases with $p \le 1$. We begin with a preparation, covering Γ, say, with $\mu(\Gamma) = 1$, for given $L \in \mathbb{N}$ by d-sets Γ_l such that

$$\Gamma = \bigcup_{l=L}^{L'} \Gamma_l, \quad \mu(\Gamma_l) \sim l^{-1}, \quad \sum_{l=L}^{L'} \mu(\Gamma_l) \sim \mu(\Gamma) = 1, \tag{6.269}$$

where $L' \in \mathbb{N}$ with $L' > L$ is chosen appropriately. This can be done as follows. For given $l \in \mathbb{N}$ and appropriately chosen large $k \in \mathbb{N}$ (in dependence on l) one finds $\sim l^{-1}\, 2^{kd}$ balls centred at Γ, of radius $\sim 2^{-k}$ and having pairwise distance of at least $\sim 2^{-k}$, such that the intersection of Γ with the union of these balls is a sub-d-set Γ^l of Γ with $\mu(\Gamma^l) \sim l^{-1}$. Now one can start for given $L \in \mathbb{N}$ with $\Gamma_L = \Gamma^L$ and applies afterwards the above procedure to $\overline{\Gamma \setminus \Gamma_L}$ and $l = L + 1$. Iteration gives the desired decomposition. This can be done in such a way that there are functions $\psi_l \in D(\mathbb{R}^n)$, $\psi_l \ge 0$,

$$\sum_{l=L}^{L'} \psi_l(\gamma) = 1 \text{ if } \gamma \in \Gamma, \quad \Gamma_l \subset \operatorname{supp} \psi_l \subset \{y \in \mathbb{R}^n : \operatorname{dist}(y, \Gamma_l) < \varepsilon_l\} \tag{6.270}$$

for some $\varepsilon_l > 0$. Let for given $l \in \mathbb{N}$ between L and L' and appropriately chosen $j(l) \in \mathbb{N}$,

$$\sum_{m=1}^{M_{j(l)}} \varphi_{j(l),m}(x) = 1 \text{ near } \Gamma, \quad 0 \le \varphi_{j(l),m} \in D(B_{j(l),m}), \tag{6.271}$$

as in (6.261), (6.262) with $M_{j(l)} \sim 2^{j(l)d}$. Let

$$j(L) < \cdots < j(l) < j(l+1) < \cdots < j(L') \tag{6.272}$$

and in analogy to (6.264),

$$\varphi^L(x) = \sum_{l=L}^{L'} \psi_l(x)\, 2^{-\frac{j(l)d}{p}} \sum_{m=1}^{M_{j(l)}} 2^{\frac{j(l)d}{p}}\, \varphi_{j(l),m}(x), \quad x \in \mathbb{R}^n. \tag{6.273}$$

First we assume that $\frac{n-d}{p} = s > \sigma_p$. Then one does not need moment conditions in Theorem 1.7, and (6.273) with large $j(l)$ is an atomic decomposition which can be written as

$$\varphi^L(x) = \sum_{l=L}^{L'} 2^{-\frac{j(l)d}{p}} \sum_{m=1}^{M'_{j(l)}} \tilde{\varphi}_{j(l),m}(x), \quad x \in \mathbb{R}^n, \tag{6.274}$$

with

$$M'_{j(l)} \sim \mu(\Gamma_l)\, 2^{j(l)d} \sim l^{-1}\, 2^{j(l)d}, \tag{6.275}$$

counting only non-vanishing terms, where the equivalence constants are independent of L. We have $\varphi^L(x) = 1$ near Γ. Then one obtains by Theorem 1.7 (i) for $q > p$ that

$$\left\| \varphi^L \,|B_{pq}^{\frac{n-d}{p}}(\mathbb{R}^n)\right\|^q \le c \sum_{l=L}^{L'} 2^{-j(l)\frac{dq}{p}} \Big(\sum_{m=1}^{M'_{j(l)}} 1 \Big)^{q/p} \le c' \sum_{l=L}^{\infty} l^{-q/p} \sim L^{1-\frac{q}{p}}. \tag{6.276}$$

This is the counterpart of (6.265). Now one can argue in the same way as in Step 3. This proves (6.256) provided that one does not need moment conditions in Theorem 1.7. But otherwise one can rely on the same arguments and references as in Step 4. This proves (6.256) also in the remaining cases. □

According to (6.242), (6.243) there is a perfect counterpart of Proposition 6.64 with the hyper-plane $\mathbb{R}^d$, $n > d \in \mathbb{N}$, in place of Γ. Similarly one can replace the target space $L_p(\Gamma)$ in Definition 6.66 by $L_p(\mathbb{R}^d)$. Then one obtains the following assertion.

Corollary 6.69. *Let $n > d \in \mathbb{N}$. Then*

$$\mathbb{D}\big(B_p(\mathbb{R}^n), L_p(\mathbb{R}^d)\big) = \begin{cases} \big(\frac{n-d}{p}, 1\big) & \text{if } p > 1, \\ \big(\frac{n-d}{p}, p\big) & \text{if } p \le 1, \end{cases} \tag{6.277}$$

and

$$\mathbb{D}\big(F_p(\mathbb{R}^n), L_p(\mathbb{R}^d)\big) = \begin{cases} \big(\frac{n-d}{p}, 0\big) & \text{if } p > 1, \\ \big(\frac{n-d}{p}, \infty\big) & \text{if } p \le 1. \end{cases} \tag{6.278}$$

Proof. This is a by-product of the proof of Theorem 6.68. On the one hand one has (6.242), (6.243). On the other hand the approximation of $g \in D(\mathbb{R}^n)$ by functions from $D_\Gamma = D(\mathbb{R}^n \setminus \Gamma)$ now with $\Gamma = \mathbb{R}^d$ is a local matter and covered by the above arguments. □

Remark 6.70. In Remark 6.65 one finds relevant references as far as traces are concerned. Density and trace assertions for the Sobolev spaces $H^s_p(\mathbb{R}^n)$ in (1.17) with $s > 0$, $1 < p < \infty$, and for the classical Besov spaces in (1.23), (1.24) with $s > 0$, $1 < p < \infty$, $1 \le q \le \infty$, are known since the late 1960s and early 1970s. Corresponding formulations and detailed references may be found in [T78], Section 2.9.4, pp. 223–26, with [Tri73] as our own contribution at this time. Density problems and related approximations for Sobolev spaces $H^s_p(\mathbb{R}^n)$ with respect to arbitrary sets Γ have been considered in [AdH96], Chapters 9, 10, based on capacity arguments and the so-called spectral synthesis. This has been extended in [HeN07] to general spaces of B-type and F-type. We followed here essentially [Tri07d].

6.4.4 Negative smoothness

For any $0 < p < \infty$ and $0 < s < n/p$ there is a number d with $0 < d < n$ and $s = \frac{n-d}{p}$. Hence the corresponding spaces maybe breaking points in Theorem 6.68. If $s < 0$ then the situation is different.

Proposition 6.71. *Let Γ be a compact set in $\mathbb{R}^n$ with Lebesgue measure $|\Gamma| = 0$. Then $D_\Gamma = D(\mathbb{R}^n \setminus \Gamma)$ is dense in all spaces*

$$A^s_{pq}(\mathbb{R}^n) \quad \textit{with } s < 0,\ 0 < p < \infty,\ 0 < q < \infty. \tag{6.279}$$

Proof. Obviously, D_Γ is dense in $L_p(\mathbb{R}^n)$ with $1 < p < \infty$. Since $D(\mathbb{R}^n)$ is dense in $A^s_{pq}(\mathbb{R}^n)$ it is sufficient to approximate $f \in D(\mathbb{R}^n)$ in $A^s_{pq}(\mathbb{R}^n)$ by functions belonging to D_Γ. Using

$$L_p(\mathbb{R}^n) \hookrightarrow A^s_{pq}(\mathbb{R}^n) \quad \text{if } s < 0,\ 1 < p < \infty,\ 0 < q < \infty, \tag{6.280}$$

one obtains the desired assertion from the density of D_Γ in $L_p(\mathbb{R}^n)$. For any ball K in $\mathbb{R}^n$ (and $s \in \mathbb{R}$, $0 < q < \infty$) there is a number $c_K > 0$ such that

$$\| f \,|A^s_{p_1,q}(\mathbb{R}^n)\| \le c_K \, \| f \,|A^s_{p_2,q}(\mathbb{R}^n)\|, \quad 0 < p_1 \le p_2 < \infty, \tag{6.281}$$

for all $f \in A^s_{p_2,q}(\mathbb{R}^n)$ with $\operatorname{supp} f \subset K$. This well-known assertion follows from Hölder's inequality and characterisations of the spaces $A^s_{pq}(\mathbb{R}^n)$ in terms of local means, [T92], Sections 2.4.6, 2.5.3, pp. 122, 138. Then the density assertion for the spaces in (6.280) can be extended to all spaces in (6.279). □

6.4.5 Curiosities

Any Radon measure μ with (6.216) can be interpreted as a tempered distribution $\mu \in S'(\mathbb{R}^n)$. This applies also to $g \in L_r(\Gamma, \mu)$ with $r \ge 1$ interpreted as in (6.219). However if $r < 1$ then the situation is totally different and the spaces $S'(\mathbb{R}^n)$ and $L_r(\Gamma, \mu)$ have nothing in common. On the other hand, Definition 6.61 makes sense and we have Proposition 6.64. But this has some curious consequences. Let Γ be a compact d-set in $\mathbb{R}^n$ with $0 < d < n$ furnished with the Hausdorff measure $\mu = \mathcal{H}^d_\Gamma$. Then it follows from Proposition 6.64 that the trace

$$\operatorname{tr}_\Gamma \colon A^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma), \quad 0 < p < 1,\ \tfrac{n-d}{p} < s < n\big(\tfrac{1}{p} - 1\big) \tag{6.282}$$

makes sense, although there are singular distributions which are elements of these spaces $A^s_{pq}(\mathbb{R}^n)$; for example, the δ-distribution belongs to $B^{n(\frac{1}{p}-1)}_{p,\infty}(\mathbb{R}^n)$ for all $0 < p \le \infty$. What does this mean for the trace of these distributions as elements of $L_p(\Gamma)$? In this context it is also reasonable to ask how to understand the traces,

$$\operatorname{tr} \colon A^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^m), \quad 0 < p < 1,\ \tfrac{n-m}{p} < s < n\big(\tfrac{1}{p} - 1\big), \tag{6.283}$$

on hyper-planes $\mathbb{R}^m \subset \mathbb{R}^n$ with $n \ge m \in \mathbb{N}$ admitting $n = m$, where (6.242), (6.243) with $d = m$ may be considered as limiting cases. According to [T06], pp. 96–97, Proposition 7.32, and its proof, pp. 315–16 , one has for a compact d'-set Γ in $\mathbb{R}^n$ with $0 \le d' < n$ and its Hausdorff measure $\mathcal{H}^{d'}_\Gamma$,

$$\mu = \mathcal{H}^{d'}_\Gamma \in B^{(n-d')(\frac{1}{p}-1)}_{p\infty}(\mathbb{R}^n), \quad |\Gamma| = 0, \ 0 < p \le \infty, \tag{6.284}$$

admitting $d' = 0$ for the δ-distribution $\mu = \delta$ with $\Gamma = \{0\}$. In particular one finds in all spaces with (6.282) and (6.283) singular distributions f with $|\operatorname{supp} f| = 0$.

Proposition 6.72. (i) *If* $s > 0$, $0 < p < \infty$, $0 < q < \infty$, *then*

$$\operatorname{tr}\colon A^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \tag{6.285}$$

exists. If, in addition, $0 < p < 1$, $0 < s \le n\big(\frac{1}{p} - 1\big)$, *then*

$$\operatorname{tr} f = 0 \text{ in } L_p(\mathbb{R}^n) \text{ for any } f \in A^s_{pq}(\mathbb{R}^n) \text{ with } |\operatorname{supp} f| = 0. \tag{6.286}$$

(ii) *Let* $0 < p < 1$. *Let* Γ *be a compact* d*-set,* $\mu = \mathcal{H}^d_\Gamma$, *and* Γ' *be a compact* d'*-set,* $\mu' = \mathcal{H}^{d'}_{\Gamma'}$ *with*

$$0 \le d' < d < n, \quad d - d' > p(n - d'). \tag{6.287}$$

Then $\mu' \in B^{\frac{n-d}{p}}_{pp}(\mathbb{R}^n)$,

$$\operatorname{tr}_\Gamma \mu' \in L_p(\Gamma) \text{ exists and } \operatorname{tr}_\Gamma \mu' = 0 \tag{6.288}$$

in $L_p(\Gamma)$.

Proof. Step 1. We prove part (i). The case $p \ge 1$ is well known. Let $0 < p \le 1$. Then one has by Definition 1.1 that

$$\begin{aligned} \|f\,|L_p(\mathbb{R}^n)\|^p &= \Big\| \sum_{j=0}^\infty (\varphi_j \hat{f})^\vee |L_p(\mathbb{R}^n)\Big\|^p \\ &\le \sum_{j=0}^\infty \|(\varphi_j \hat{f})^\vee\,|L_p(\mathbb{R}^n)\|^p \le c\,\|f\,|B^s_{pq}(\mathbb{R}^n)\|^p. \end{aligned} \tag{6.289}$$

One obtains

$$\|\varphi\,|L_p(\mathbb{R}^n)\| \le c\,\|\varphi\,|A^s_{pq}(\mathbb{R}^n)\|, \quad s > 0, \ 0 < p < \infty, \ 0 < q < \infty, \tag{6.290}$$

for all $\varphi \in S(\mathbb{R}^n)$ as requested in Definition 6.61. This proves (6.285). Let, in addition,

$$f \in A^s_{pq}(\mathbb{R}^n) \quad \text{with } \Gamma = \operatorname{supp} f \text{ compact}, |\Gamma| = 0, \tag{6.291}$$

be non-trivial (hence $0 < p < 1$, $0 < s \le n\big(\frac{1}{p} - 1\big)$) and let

$$\Gamma_\varepsilon = \{y \in \mathbb{R}^n : \operatorname{dist}(y, \Gamma) < \varepsilon\} \tag{6.292}$$

for $\varepsilon > 0$. Then f can be approximated in $A^s_{pq}(\mathbb{R}^n)$ by $D(\mathbb{R}^n)$-functions with support in Γ_ε (multiplying an approximating sequence of $D(\mathbb{R}^n)$-functions with a suitable function from $D(\Gamma_\varepsilon)$). This proves $\operatorname{tr} f \subset \Gamma_\varepsilon$ for any $\varepsilon > 0$. Hence $\operatorname{tr} f \subset \Gamma$ and $\operatorname{tr} f = 0$ in $L_p(\mathbb{R}^n)$. This can be extended to all f covered by (6.286).

Step 2. We prove part (ii). First we remark that (6.287) and (6.284) ensure that

$$(n - d')\Big(\frac{1}{p} - 1\Big) > \frac{n-d}{p} \quad \text{and} \quad \mu' \in B_{pp}^{\frac{n-d}{p}}(\mathbb{R}^n). \tag{6.293}$$

Then one obtains by Proposition 6.64 that $\operatorname{tr}_\Gamma \mu' \in L_p(\Gamma)$ exists. Let

$$\Gamma' \subset \Gamma'_j = \bigcup_{m=1}^{M_j} B\big(\gamma'_{j,m}, 2^{-j}\big), \quad M_j \sim 2^{jd'}, \tag{6.294}$$

where $B(\gamma'_{j,m}, 2^{-j})$ are open balls centred at Γ' and of radius 2^{-j} with $j \in \mathbb{N}$. One can approximate μ' in $B_{pp}^{\frac{n-d}{p}}(\mathbb{R}^n)$ by $D(\mathbb{R}^n)$-functions with support in Γ'_j (similarly as indicated above). However,

$$\mu(\Gamma'_j) \le c\, 2^{-jd}\, 2^{jd'} \to 0 \quad \text{if } j \to \infty. \tag{6.295}$$

This proves $\operatorname{tr}_\Gamma \mu' = 0$ in $L_p(\Gamma)$. □

Remark 6.73. The above proposition shows that (tempered) distributions in $\mathbb{R}^n$ on the one hand and measurable functions on the other hand are rather different worlds and that great care is required when crossing the border. For the δ-distribution one has (6.284) with $\mu = \delta$ and $d' = 0$. Then one obtains by (6.286) that

$$\operatorname{tr} \delta = 0 \quad \text{in } L_p(\mathbb{R}^n) \text{ with } 0 < p < 1, \tag{6.296}$$

what looks somewhat curious.

6.4.6 Pointwise evaluation

In Definition 6.66 we excluded $d = 0$ and $d = n$. Whereas $d = n$ does not fit in the above scheme (with exception of a few curiosities discussed in Section 6.4.5) the case $d = 0$ makes sense especially when Γ consists of finitely many points in $\mathbb{R}^n$,

$$\Gamma = \big\{x^l \in \mathbb{R}^n : l = 1, \dots, L\big\}, \quad L \in \mathbb{N},\ n \in \mathbb{N}. \tag{6.297}$$

The trace problem reduces to the question of whether the embedding

$$\mathrm{id}\colon A^s_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \tag{6.298}$$

is continuous. As before, $C(\mathbb{R}^n)$ is the space of all complex-valued continuous bounded functions in $\mathbb{R}^n$, normed by

$$\|f\,|C(\mathbb{R}^n)\| = \sup_{x\in\mathbb{R}^n} |f(x)|, \tag{6.299}$$

whereas $A^s_{pq}(\mathbb{R}^n)$ with $A \in \{B, F\}$ has the same meaning as in Definition 1.1. For the embedding (6.298) one has the following final answer,

$$B^s_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \quad \text{if, and only if,} \quad \begin{cases} 0 < p \le \infty,\ 0 < q \le \infty,\ s > n/p, \\ 0 < p \le \infty,\ 0 < q \le 1,\ s = n/p, \end{cases} \tag{6.300}$$

and

$$F^s_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \quad \text{if, and only if,} \quad \begin{cases} 0 < p < \infty,\ 0 < q \le \infty,\ s > n/p, \\ 0 < p \le 1,\ 0 < q \le \infty,\ s = n/p. \end{cases} \tag{6.301}$$

This is well known. One may consult [T01], Section 11, and [Har07], Section 7.2, where one finds also the necessary references. We only mention that the sharp assertion in the limiting case $s = n/p$ goes back to [SiT95]. This is the counterpart of Proposition 6.64. The difference between (6.240) and (6.300) is also reflected by what follows. The counterpart of Definition 6.66 reads now as follows. As before we put $D_\Gamma = D(\mathbb{R}^n\setminus\Gamma)$.

Definition 6.74. Let $n \in \mathbb{N}$ and $0 < p < \infty$. Let

$$A_p(\mathbb{R}^n) = \big\{A^s_{pq}(\mathbb{R}^n) :\ 0 < q < \infty,\ s \in \mathbb{R}\big\}. \tag{6.302}$$

Let Γ be as in (6.297). Let $\sigma \in \mathbb{R}$. Then

$$\mathbb{D}\big(A_p(\mathbb{R}^n), \Gamma\big) = (\sigma, u) \quad \text{with } 0 < u < \infty \tag{6.303}$$

is called the dichotomy of $\{A_p(\mathbb{R}^n), \Gamma\}$ if

$$\mathrm{id}\colon A^s_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \quad \text{for} \begin{cases} s > \sigma, & 0 < q < \infty, \\ s = \sigma, & 0 < q \le u, \end{cases} \tag{6.304}$$

and

$$D_\Gamma \quad \text{is dense in} \quad A^s_{pq}(\mathbb{R}^n) \quad \text{for} \begin{cases} s = \sigma, & u < q < \infty, \\ s < \sigma, & 0 < q < \infty. \end{cases} \tag{6.305}$$

Furthermore,

$$\mathbb{D}\big(A_p(\mathbb{R}^n), \Gamma\big) = (\sigma, 0) \tag{6.306}$$

means that

$$\left.\begin{aligned} &\text{id exists for } s > \sigma,\ 0 < q < \infty, \\ &D_\Gamma \text{ is dense in } A^s_{pq}(\mathbb{R}^n) \text{ for } s \le \sigma,\ 0 < q < \infty, \end{aligned}\right\} \tag{6.307}$$

and

$$\mathbb{D}\big(A_p(\mathbb{R}^n), \Gamma\big) = (\sigma, \infty) \tag{6.308}$$

means that

$$\left.\begin{array}{l} \text{id exists for } s \ge \sigma, 0 < q < \infty, \\ D_\Gamma \text{ is dense in } A^s_{pq}(\mathbb{R}^n) \text{ for } s < \sigma, 0 < q < \infty. \end{array}\right\} \tag{6.309}$$

Remark 6.75. This is the direct counterpart of Definition 6.66. The justification of the above definition is the same as in Remark 6.67. We are now not only interested in a counterpart of Theorem 6.68 with the breaking smoothness $s = n/p$, but also in the question of whether D_Γ is dense in

$$A^{s,\Gamma}_{pq}(\mathbb{R}^n) = \big\{f \in A^s_{pq}(\mathbb{R}^n) : \textstyle\sum_{l=1}^L |f(x^l)| = 0\big\} \tag{6.310}$$

for spaces $A^s_{pq}(\mathbb{R}^n)$ satisfying (6.300), (6.301). This cannot happen if one has the embedding (6.298) with $C^1(\mathbb{R}^n)$ in place of $C(\mathbb{R}^n)$, because one has in this case the same sharp criterion as in (6.300), (6.301) with $\frac{n}{p}+1$ in place of $\frac{n}{p}$, [T01], Theorem 11.4, pp. 170–71, [Har07], Section 7.2. We do not bother about the limiting cases $s = 1 + \frac{n}{p}$ and restrict the question of whether D_Γ is dense in $A^{s,\Gamma}_{pq}(\mathbb{R}^n)$ to the spaces (6.300), (6.301) with $s < 1 + \frac{n}{p}$. Recall again that $D_\Gamma = D(\mathbb{R}^n \setminus \Gamma)$.

Theorem 6.76. *Let $n \in \mathbb{N}$ and*

$$\Gamma = \big\{x^l \in \mathbb{R}^n : l = 1, \dots, L\big\}, \quad L \in \mathbb{N}. \tag{6.311}$$

(i) *Let $0 < p < \infty$. Then*

$$\mathbb{D}\big(B_p(\mathbb{R}^n), \Gamma\big) = \Big(\frac{n}{p}, 1\Big) \tag{6.312}$$

and

$$\mathbb{D}\big(F_p(\mathbb{R}^n), \Gamma\big) = \begin{cases} \big(\frac{n}{p}, 0\big) & \text{if } p > 1, \\ \big(\frac{n}{p}, \infty\big) & \text{if } p \le 1. \end{cases} \tag{6.313}$$

(ii) *Let $0 < p < \infty$, $0 < q < \infty$ and $s < 1 + \frac{n}{p}$. Then D_Γ is dense in $A^{s,\Gamma}_{pq}(\mathbb{R}^n)$ with* (6.300), (6.301).

Proof. Step 1. It is sufficient to deal with $\Gamma = \{0\}$. We begin with a preparation and approximate $f \in D(\mathbb{R}^n)$ in $B^{n/p}_{pq}(\mathbb{R}^n)$ with $0 < p < \infty$, $1 < q < \infty$ by functions $f^J \in D(\mathbb{R}^n)$ with $f^J(0) = 0$. Basically we rely on the same construction as in Step 2 of the proof of Theorem 6.68. For $2 \le J < \infty$ let $J' \in \mathbb{N}$ be such that

$$\sum_{j=J}^{J'+1} r_j = 1 \quad \text{with } r_j = j^{-1} \text{ if } J \le j \le J' \text{ and } 0 < r_{J'+1} \le (J'+1)^{-1}. \tag{6.314}$$

Let $0 \le \psi \in D(\mathbb{R}^n)$ with

$$\psi(x) > 0 \quad \text{if, and only if,} \quad |x| \le 1, \quad \psi(0) = 1. \tag{6.315}$$

Let

$$f_J(x) = \sum_{j=J}^{J'+1} r_j\, \psi(2^j x)\, f(x), \quad x \in \mathbb{R}^n. \tag{6.316}$$

Then it follows from Theorem 1.7 and Definition 1.5 that

$$\| f_J \,| B^{n/p}_{pq}(\mathbb{R}^n)\|^q \le c \sum_{j=J}^{J'+1} j^{-q} \sim J^{1-q}. \tag{6.317}$$

Since $q > 1$ one obtains that $f^J = f - f_J$ approximates f in $B^{n/p}_{pq}(\mathbb{R}^n)$. Furthermore $f^J(0) = 0$. To justify a corresponding approximation in $F^{n/p}_{pq}(\mathbb{R}^n)$ with $p > 1$, $0 < q < \infty$, we recall that one has the embedding

$$B^{n/p_0}_{p_0 u}(\mathbb{R}^n) \hookrightarrow F^{n/p}_{pq}(\mathbb{R}^n) \hookrightarrow B^{n/p_1}_{p_1 v}(\mathbb{R}^n), \quad 0 < p_0 < p < p_1 < \infty, \tag{6.318}$$

if, and only if, $0 < u \le p \le v \le \infty$. This may be found in [ET96], p. 44, with the reference to the original papers [Fra86], [Jaw77], [SiT95]. There is also a new proof based on wavelets in [Vyb08b]. In particular one may choose $u = p > 1$. Then one obtains the desired approximation in $F^{n/p}_{pq}(\mathbb{R}^n)$ with $p > 1$, $0 < q < \infty$, from (6.318) and the above approximation in $B^{n/p_0}_{p_0 p}(\mathbb{R}^n)$. Recall that $D(\mathbb{R}^n)$ is dense in all spaces $A^s_{pq}(\mathbb{R}^n)$ with $p < \infty$, $0 < q < \infty$. Then it follows from the above considerations and elementary embeddings that

$$\big\{ f \in D(\mathbb{R}^n) : \ f(0) = 0 \big\} \text{ is dense in all spaces } A^s_{pq}(\mathbb{R}^n),\ p < \infty,\ q < \infty, \tag{6.319}$$

which are not covered by (6.300), (6.301).

Step 2. We prove both parts of the theorem. Let $f \in A^s_{pq}(\mathbb{R}^n)$ with $p < \infty$, $q < \infty$ in (6.300), (6.301) and $f(0) = 0$. Then f can be approximated in $A^s_{pq}(\mathbb{R}^n)$ by functions $f_\varepsilon \in D(\mathbb{R}^n)$ with $f_\varepsilon(0) \to 0$ if $\varepsilon \to 0$. This is the same situation as in the proof of Proposition 6.21. One has similarly as there that the functions $f \in D(\mathbb{R}^n)$ with $f(0) = 0$ are dense in $A^{s,\Gamma}_{pq}(\mathbb{R}^n)$ where $\Gamma = \{0\}$. Then one obtains by Step 1 that it is sufficient to approximate a function

$$f \in D(\mathbb{R}^n), \quad \operatorname{supp} f \subset \{x \in \mathbb{R}^n : \ |x| < 1\}, \quad f(0) = 0, \tag{6.320}$$

in

$$B^s_{pq}(\mathbb{R}^n), \quad 0 < p < \infty,\ 0 < q < \infty,\ n\big(\tfrac{1}{p} - 1\big)_+ < s < 1 + \tfrac{n}{p}, \tag{6.321}$$

by functions $g \in D(\mathbb{R}^n)$ with $0 \notin \operatorname{supp} g$. Let

$$\varphi \in D(\mathbb{R}^n), \quad \operatorname{supp} \varphi \subset \big\{ x \in \mathbb{R}^n : \ \tfrac{1}{2} < |x| < 2 \big\} \tag{6.322}$$

such that

$$\sum_{j=0}^{\infty} \varphi(2^j x) = 1 \quad \text{if } 0 < |x| < 1. \tag{6.323}$$

Then

$$f(x) = \sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})-j} \left[2^{-j(s-\frac{n}{p})} \, 2^j \, \varphi(2^j x) \, f(x) \right] \tag{6.324}$$

is an atomic decomposition in $B^s_{pq}(\mathbb{R}^n)$ with (6.321). We refer again to Theorem 1.7 (no moment conditions are needed). The extra factor 2^j comes from $|f(x)| \le c\,|x|$. Let f^J be given by (6.324) with $j \ge J$ in place of $j \in \mathbb{N}_0$. One has for given $\varepsilon > 0$ that

$$\| f^J \,|B^s_{pq}(\mathbb{R}^n)\|^q \le \sum_{j=J}^{\infty} 2^{j(s-\frac{n}{p}-1)q} \le \varepsilon \tag{6.325}$$

if $J \ge J(\varepsilon)$. Then $g = f - f^J$ is the desired approximation. □

6.4.7 A comment on sampling numbers

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition 3.4 (iii) if $n \ge 2$ or a bounded interval in $\mathbb{R}$ if $n = 1$. There is a more or less obvious counterpart of Theorem 6.76 with Ω in place of $\mathbb{R}^n$. But it seems to be reasonable to fix some related notation. Recall that we introduced the spaces $A^s_{pq}(\Omega)$ in Definition 2.1 (i) as the restriction of $A^s_{pq}(\mathbb{R}^n)$ to Ω and the spaces $\mathring{A}^s_{pq}(\Omega)$ in Definition 5.17 (i) as the completion of $D(\Omega)$ in $A^s_{pq}(\Omega)$. Let $C(\Omega)$ be the restriction of $C(\mathbb{R}^n)$ to Ω. This coincides with $C(\Omega) = C^0(\Omega)$ in Definition 5.17 (ii). Then one has

$$\mathrm{id}\colon \; A^s_{pq}(\Omega) \hookrightarrow C(\Omega) \tag{6.326}$$

for the same parameters as in (6.300), (6.301). With

$$\Gamma = \left\{ x^l \in \Omega : \; l = 1, \ldots, L \right\}, \quad L \in \mathbb{N}, \; n \in \mathbb{N}, \tag{6.327}$$

one has for the spaces in (6.326) the counterpart

$$A^{s,\Gamma}_{pq}(\Omega) = \left\{ f \in A^s_{pq}(\Omega) : \; \textstyle\sum_{l=1}^{L} |f(x^l)| = 0 \right\} \tag{6.328}$$

of (6.310). Both parts of Theorem 6.76 with Ω in place of $\mathbb{R}^n$ and the restriction

$$D^{\Omega}_{\Gamma} = D_\Gamma | \Omega \quad \text{of } D_\Gamma = D(\mathbb{R}^n \setminus \Gamma) \tag{6.329}$$

in place of D_Γ remain valid. One can also replace there (in obvious notation)

$$A^s_{pq}(\Omega) \text{ by } \mathring{A}^s_{pq}(\Omega), \quad A^{s,\Gamma}_{pq}(\Omega) \text{ by } \mathring{A}^{s,\Gamma}_{pq}(\Omega), \quad \text{and} \quad D^{\Omega}_{\Gamma} \text{ by } D(\Omega \setminus \Gamma), \tag{6.330}$$

where now $p = \infty$ and/or $q = \infty$ are admitted.

Problems of sampling, tractability and optimal recovery of functions are very fashionable subjects nowadays. They have been studied by many authors. As for the background of this theory we refer to [TWW88], [Nov88], [NoW08]. Sampling numbers rely on *standard information*, hence methods which are based on function values. They have a substantial history. Related references may be found in [NoT04], Remark 24, p. 341, and [T06], Remark 4.42, p. 228, which will not be repeated here. We do not deal systematically with sampling numbers. We are only interested in the connection between sampling numbers and the Ω-version of Theorem 6.76 described above. But first we recall what is meant by (non-linear) sampling numbers in the context of the above spaces $A^s_{pq}(\Omega)$. We assume that the source space $A^{s_1}_{p_1q_1}(\Omega)$ satisfies (6.326), hence

$$\mathrm{id}\colon A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow C(\Omega), \tag{6.331}$$

which means that s_1, p_1, q_1 are restricted by the right-hand sides of (6.300), (6.301). Let

$$\mathrm{id}\colon A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow A^{s_2}_{p_2q_2}(\Omega) \quad \text{be compact,} \tag{6.332}$$

which is equivalent to

$$s_1 > s_2 + n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+, \qquad 0 < q_1, q_2 \le \infty. \tag{6.333}$$

Recall that $a_+ = \max(0,a)$ where $a \in \mathbb{R}$. With Γ as in (6.327) the *information map*,

$$N_L\colon A^{s_1}_{p_1q_1}(\Omega) \mapsto \mathbb{C}^L, \quad L \in \mathbb{N}, \tag{6.334}$$

given by

$$N_L f = \big(f(x^1), \ldots, f(x^L)\big), \quad f \in A^{s_1}_{p_1q_1}(\Omega), \tag{6.335}$$

makes sense. As usual, $\mathbb{C}^L$ is the collection of all L-tuples of complex numbers. Let

$$S_L = \Phi_L \circ N_L \quad \text{with } \Phi_L\colon \mathbb{C}^L \mapsto A^{s_2}_{p_2q_2}(\Omega) \tag{6.336}$$

be an arbitrary map (also called *method* or *algorithm*), hence

$$S_L f = \Phi_L\big(f(x^1), \ldots, f(x^L)\big) \in A^{s_2}_{p_2q_2}(\Omega), \quad f \in A^{s_1}_{p_1q_1}(\Omega). \tag{6.337}$$

One wishes to recover a given function $f \in A^{s_1}_{p_1q_1}(\Omega)$ in $A^{s_2}_{p_2q_2}(\Omega)$ by asking for optimally scattered sampling points Γ and optimally chosen methods Φ_L. With id as in (6.332),

$$\widetilde{g_L}(\mathrm{id}) = \inf\Big[\sup\big\{\|f - S_L f\,|A^{s_2}_{p_2q_2}(\Omega)\| : \|f\,|A^{s_1}_{p_1q_1}(\Omega)\| \le 1\big\}\Big] \tag{6.338}$$

is the Lth sampling number, where the infimum is taken over all L-tuples $\{x^l\}_{l=1}^L \subset \Omega$ and all maps $S_L = \Phi_L \circ N_L$ according to (6.334)–(6.337). This definition goes back to [NoT04], Definition 17, p. 339, and may also be found in [T06], Definition 4.32, p. 219. It can be (equivalently) reformulated as follows.

Definition 6.77. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition 3.4 (iii) if $n \geq 2$ or a bounded interval in $\mathbb{R}$ if $n = 1$. Let $\Gamma = \{x^l\}_{l=1}^L \subset \Omega$. Let

$$s_1 \in \mathbb{R},\ s_2 \in \mathbb{R},\ \text{and}\ 0 < p_1, p_2 \leq \infty\ (< \infty \text{ for } F\text{-spaces}),\ 0 < q_1, q_2 \leq \infty \tag{6.339}$$

with

$$s_1 > s_2 + n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+, \tag{6.340}$$

and let

$$\mathrm{id}\colon A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow A^{s_2}_{p_2q_2}(\Omega). \tag{6.341}$$

(i) Let, in addition,

$$A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow C(\Omega) \tag{6.342}$$

and

$$A^{s_1,\Gamma}_{p_1q_1}(\Omega) = \big\{ f \in A^{s_1}_{p_1q_1}(\Omega) : \textstyle\sum_{l=1}^L |f(x^l)| = 0 \big\}. \tag{6.343}$$

Then for $k \in \mathbb{N}$,

$$g_k(\mathrm{id}) = \inf \big[\sup \big\{ \|f\,|A^{s_2}_{p_2q_2}(\Omega)\| : \|f\,|A^{s_1}_{p_1q_1}(\Omega)\| \leq 1,\ f \in A^{s_1,\Gamma}_{p_1q_1}(\Omega) \big\} \big] \tag{6.344}$$

where the infimum is taken over all Γ with $\operatorname{card}\Gamma \leq k$.

(ii) Then for $k \in \mathbb{N}$,

$$g^k(\mathrm{id}) = \inf \big[\sup \big\{ \|f\,|A^{s_2}_{p_2q_2}(\Omega)\| : \|f\,|A^{s_1}_{p_1q_1}(\Omega)\| \leq 1,\ f \in D^{\Omega}_{\Gamma} \big\} \big] \tag{6.345}$$

where the infimum is taken over all Γ with $\operatorname{card}\Gamma \leq k$.

Remark 6.78. As remarked in (6.332), (6.333) the embedding (6.341) makes sense. It is compact, if and only if, one has (6.340). This justifies the definitions of $g_k(\mathrm{id})$ and $g^k(\mathrm{id})$. According to [NoT04], Proposition 19, p. 340, or [T06], Proposition 4.34, p. 220, one has

$$\widetilde{g_k}(\mathrm{id}) \sim g_k(\mathrm{id}), \quad k \in \mathbb{N}, \tag{6.346}$$

where the original (non-linear) sampling numbers $\widetilde{g_k}(\mathrm{id})$ are given by (6.338). In contrast to the sampling numbers $g_k(\mathrm{id})$ in part (i) the corresponding numbers in (6.345) make sense for any compact embedding (6.341). How they are related and are the numbers $g^k(\mathrm{id})$ of any use? We do not deal with this question systematically. We have a closer look at a model case, replacing the target space $A^{s_2}_{p_2q_2}(\Omega)$ by $L_1(\Omega)$, hence

$$\mathrm{id}\colon A^s_{pq}(\Omega) \hookrightarrow L_1(\Omega). \tag{6.347}$$

Characterisations for which $A \in \{B, F\}$ and s, p, q the embedding (6.347) is continuous may be found in [T01], Theorem 11.2, pp. 168–69, and [Har07], pp. 111–12, with a reference to [SiT95]. It follows that id in (6.347) is compact if, and only if,

$$0 < p \leq \infty\ (p < \infty \text{ for } F\text{-spaces}),\ 0 < q \leq \infty,\ s > \sigma_p = n\big(\tfrac{1}{p} - 1\big)_+. \tag{6.348}$$

Theorem 6.79. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ according to Definition* 3.4 (iii) *if $n \ge 2$ or a bounded interval in $\mathbb{R}$ if $n = 1$. Let $g_k(\mathrm{id})$ and $g^k(\mathrm{id})$ be the sampling numbers according to Definition* 6.77 *with respect to the compact embedding* (6.347) (*or equivalently* (6.348)) *in place of* (6.341).

(i) *Let, in addition,*

$$A^s_{pq}(\Omega) \hookrightarrow C(\Omega) \tag{6.349}$$

(*or equivalently the right-hand sides of* (6.300), (6.301)) *and $s < 1 + \frac{n}{p}$. Then*

$$g^k(\mathrm{id}) \sim g_k(\mathrm{id}) \sim k^{-\frac{s}{n}+(\frac{1}{p}-1)_+}, \quad k \in \mathbb{N}. \tag{6.350}$$

(ii) *Let, in addition,* (6.349) *not be valid. Then*

$$g^k(\mathrm{id}) \sim 1, \quad k \in \mathbb{N}. \tag{6.351}$$

Proof. Step 1. We prove part (i). If $p < \infty$, $q < \infty$ then the first equivalence in (6.350) follows from Theorem 6.76 (ii) transferred to Ω as indicated at the beginning of this Section 6.4.7. If $s > n/p$ then the second equivalence in (6.350) is a special case of [NoT04], Theorem 23, p. 341, or [T06], Theorem 4.37, p. 224. But it is valid for all spaces in (6.349). This is not immediately covered by [NoT04], [T06]. We add a respective comment in Remark 6.80 below. It remains to extend these assertions to spaces with $p = \infty$ and/or $q = \infty$. Let

$$\mathrm{id}_0 : \mathring{A}^s_{pq}(\Omega) \hookrightarrow L_1(\Omega). \tag{6.352}$$

The different behaviour of id in (6.347) and of id_0 in (6.352) near the boundary of Ω is immaterial. But otherwise one can now extend the above arguments to $p = \infty$ and/or $q = \infty$ as indicated in (6.330). In particular, by (6.345)

$$g^k(\mathrm{id}) = g^k(\mathrm{id}_0) \sim g_k(\mathrm{id}_0), \quad k \in \mathbb{N}. \tag{6.353}$$

By the proof in [NoT04], [T06] one obtains that

$$g_k(\mathrm{id}_0) \sim k^{-\frac{s}{n}+(\frac{1}{p}-1)_+} \sim g_k(\mathrm{id}), \quad k \in \mathbb{N}. \tag{6.354}$$

This proves (6.350) also if $p = \infty$ and /or $q = \infty$.

Step 2. We prove part (ii). If (6.349) is not valid then $p < \infty$. If, in addition, $q < \infty$ then (6.351) follows from (6.345) and the Ω-version of Theorem 6.76 saying that D^Ω_Γ is dense in $A^s_{pq}(\Omega)$. Spaces with $q = \infty$ can now be incorporated by monotonicity arguments since one has already (6.351) for suitable smaller spaces. □

Remark 6.80. We add a comment about the second equivalence in (6.350) for spaces $A^s_{pq}(\Omega)$ with (6.349) and $s = n/p$. First we remark that one can extend [T06], Proposition 4.36(i), p. 222, to the spaces $F^s_{pq}(\Omega)$ with $0 < p \le 1$, $0 < q \le \infty$, $s = n/p$, according to (6.301). As there one may assume that $q \ge p$. The arguments rely on [T06], Corollary 4.13(ii), p. 202, which can be extended to F^s_{pq} with $0 < p \le 1$,

$q \geq p$, $s = n/p$. Afterwards one can apply the arguments from the proof of [T06], Theorem 4.37, p. 224, to these cases. This proves the second equivalence in (6.350) for all spaces $F^s_{pq}(\Omega)$ with (6.349). By

$$B^{n/p}_{p,q}(\Omega) \hookrightarrow B^n_{1,1}(\Omega), \quad 0 < p \leq 1,\ 0 < q \leq 1, \tag{6.355}$$

one obtains a corresponding assertion for these B-spaces in the limiting situation $s = n/p$ in (6.300). Complex interpolation between $B^n_{1,1}(\Omega)$ and $B^0_{\infty,1}(\Omega)$ extends these considerations to the spaces $B^{n/p}_{p,q}(\Omega)$ with $1 < p < \infty$, $q \geq 1$.

Remark 6.81. It is not the aim of this Section 6.4.7 to study the behaviour of (non-linear) sampling numbers g_k for diverse source and target spaces for their own sake. We wanted only to make clear how closely and also how naturally these numbers are related to the pointwise dichotomy according to Theorem 6.76. This may justify that we fixed the target space by $L_1(\Omega)$ in (6.347) and in Theorem 6.79. Although we used in the above arguments occasionally the right-hand side of (6.350) there is little doubt that these considerations can be extended to (6.332) with (6.331). Secondly we tried to find a characterisation of g_k which does not use (6.331) and which gives the possibility to extend the definition of non-linear sampling numbers g_k to source spaces which do not necessarily satisfy (6.331). These are the sampling numbers g^k according to (6.345) with (6.350). Questions of this type have attracted some attention. We refer to [Hei08]. But the outcome both in [Hei08] and also in the above Theorem 6.79 (ii) is largely negative. But it should be said that this observation is the main motivation in [Hei08] to step from deterministic approximations to randomised approximations what changes the situation.

Remark 6.82. Sampling numbers for embeddings of type (6.341), (6.342) in bounded Lipschitz domains Ω with the target spaces $L_{p_2}(\Omega)$, $0 < p_2 \leq \infty$, in place of $A^{s_2}_{p_2q_2}(\Omega)$, have been considered in [NoT04]. This has been extended in [Tri05] to target spaces $A^{s_2}_{p_2q_2}(\Omega)$ with $s_2 > \sigma_{p_2}$. One may also consult [T06], Section 4.3. In [NoT04], [Tri05], [T06] one finds references to earlier papers. To some extent one can replace bounded Lipschitz domains by arbitrary domains Ω with $|\Omega| < \infty$, or E-thick domains Ω according to Definition 3.1 (ii) with $|\Omega| < \infty$. We return to this point in Theorem 6.87 below with a reference to [Tri07b]. Sampling numbers for target spaces of type $B^0_{p_2q_2}(\Omega)$ show some curious behaviour, different from $L_{p_2}(\Omega)$, [Vyb08a]. Finally we refer to [Vyb07a] where also target spaces $A^{s_2}_{p_2q_2}(\Omega)$ with $s_2 < 0$ have been considered. Some of these recent results have also been surveyed in [NoW08], Section 4.2.4, where the non-linear sampling numbers g_k are denoted as

$$g_k = e^{\mathrm{wor}}\big(k, \Lambda^{\mathrm{std}}\big), \quad k \in \mathbb{N}. \tag{6.356}$$

As far as this notation is concerned one may consult [NoW08], Sections 4.1.1, 4.14. We only mention that Λ^{std} refers to pointwise evaluation, called *standard information* in the context of theory developed there.

6.5 Polynomial reproducing formulas

6.5.1 Global reproducing formulas

Polynomial reproducing formulas play some role in numerical analysis. They can be used to construct stable wavelet bases on intervals and on domains. We refer to [Coh03], Section 2.12, especially pp. 125–26. We commented in Section 6.4.7 on sampling numbers. Assertions as on the right-hand side of (6.350) are proved in [NoT04] and [T06], Section 4.3.3, with the help of a local polynomial reproducing formula due to H. Wendland, [Wen01]. The aim of the present Section 6.5 is rather modest. We wish to make clear how one derives polynomial reproducing formulas from wavelet expansions. We begin with corresponding assertions in $\mathbb{R}^n$.

We use the weighted spaces $A^s_{pq}(\mathbb{R}^n, w)$ as introduced in Definition 1.22, specified by

$$B^s_{pp}(\mathbb{R}^n, w_\gamma), \quad 1 \le p \le \infty,\ s > 0,\ w_\gamma(x) = (1 + |x|^2)^{\gamma/2}, \tag{6.357}$$

with $\gamma \in \mathbb{R}$ as in (1.136). According to Theorem 1.26 any $f \in B^s_{pp}(\mathbb{R}^n, w_\gamma)$ can be expanded by

$$f = \sum_{j,G,m} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^j_{G,m}, \quad \lambda \in b^s_{pp}(w_\gamma), \tag{6.358}$$

with

$$\lambda^{j,G}_m = 2^{jn/2} \int_{\mathbb{R}^n} f(x) \, \Psi^j_{G,m}(x) \, \mathrm{d}x \tag{6.359}$$

where $\Psi^j_{G,m}$ are the real wavelets (1.91) based on (1.87), (1.88). All these wavelets satisfy moment conditions of type (1.88) extended to $\mathbb{R}^n$ with exception of

$$\Psi_{F,m}(x) = \Psi^0_{G,m}(x) = \prod_{r=1}^n \psi_F(x_r - m_r), \quad m \in \mathbb{Z}^n,\ x \in \mathbb{R}^n, \tag{6.360}$$

with $G = (F, \ldots, F)$. Let $f = P$ be a polynomial of degree N and let $N < u \in \mathbb{N}$ in (1.88) and Theorem 1.26. If $P \in B^s_{pp}(\mathbb{R}^n, w_\gamma)$ then one has the expansion (6.358) where all coefficients $\lambda^{j,G}_m$ are zero with possible exception of $\lambda^{0,G}_m$ with $G = \{F, \ldots, F\}$ and $m \in \mathbb{Z}^n$, hence

$$P(x) = \sum_{m \in \mathbb{Z}^n} \left(\int_{\mathbb{R}^n} P(y) \, \Psi_{F,m}(y) \, \mathrm{d}y \right) \Psi_{F,m}(x), \quad x \in \mathbb{R}^n. \tag{6.361}$$

We justify this expansion.

Theorem 6.83. *Let P be a polynomial of degree at most N in $\mathbb{R}^n$. Let $s > 0$,*

$$1 \le p \le \infty, \quad \max(s, N) < u \in \mathbb{N}, \quad \gamma < -N - \frac{n}{p}. \tag{6.362}$$

Then $P \in B^s_{pp}(\mathbb{R}^n, w_\gamma)$, and one has (6.361), unconditional convergence being in $B^s_{pp}(\mathbb{R}^n, w_\gamma)$.

Proof. Since supp $\widehat{P} = \{0\}$ it follows from (1.117) that

$$P \in B^s_{pp}(\mathbb{R}^n, w_\gamma) \quad \text{if, and only if,} \quad w_\gamma P \in L_p(\mathbb{R}^n). \tag{6.363}$$

But this is ensured by (6.362). Furthermore we have by Theorem 1.26 and (1.119) that

$$\|P \,|B^s_{pp}(\mathbb{R}^n, w_\gamma)\| \sim \|\lambda(P) \,|b^s_{pp}(w_\gamma)\| \le c \Big(\sum_{m \in \mathbb{Z}^n} (1+|m|)^{\gamma p + Np} \Big)^{1/p} \tag{6.364}$$

with the usual modification if $p = \infty$. This show that (6.361) converges unconditionally for all p with $1 \le p \le \infty$, including $p = \infty$. □

Remark 6.84. Recall the well-known assertion $\int_{\mathbb{R}} \psi_F(x)\,dx = 1$, [T06], p. 31. Then one obtains by (6.361) with $P(x) = 1$ that

$$\sum_{m \in \mathbb{Z}^n} \Psi_{F,m}(x) = 1, \quad x \in \mathbb{R}^n, \tag{6.365}$$

is a resolution of unity. The right-hand side of (6.361) makes sense for any, say, continuous function f in $\mathbb{R}^n$, hence

$$(Uf)(x) = \sum_{m \in \mathbb{Z}^n} \Big(\int_{\mathbb{R}^n} f(y)\, \Psi_{F,m}(y)\,dy \Big)\, \Psi_{F,m}(x), \quad x \in \mathbb{R}^n. \tag{6.366}$$

Polynomials $f = P$ of degree at most N are reproduced, what explains the notation *polynomial reproducing formula*. Recall that

$$\Psi_{F,m} \in C^u(\mathbb{R}^n), \quad \operatorname{supp} \Psi_{F,m} \subset B(m,c) = \big\{ y \in \mathbb{R}^n : |y-m| < c \big\}, \tag{6.367}$$

for some $c > 0$. One can replace $\Psi_{F,m}(x)$ by suitable functions $H_m(x)$ with

$$H_m \in C^\infty(\mathbb{R}^n), \quad \operatorname{supp} H_m \subset B(m,c'), \quad m \in \mathbb{Z}^n, \tag{6.368}$$

for some $c' > 0$. Let ω be a compactly supported C^∞ function satisfying

$$\int_{\mathbb{R}^n} \omega(y)\,dy = 1, \quad \int_{\mathbb{R}^n} y^\alpha\, \omega(y)\,dy = 0, \quad 0 < |\alpha| \le N. \tag{6.369}$$

Mollification

$$(\omega \star g)(x) = \int_{\mathbb{R}^n} g(y)\,\omega(x-y)\,dy = \int_{\mathbb{R}^n} g(x-y)\,\omega(y)\,dy \tag{6.370}$$

of (6.361) results in

$$P(x) = \sum_{m \in \mathbb{Z}^n} \Big(\int_{\mathbb{R}^n} P(y)\, \Psi_{F,m}(y)\,dy \Big)\, H_m(x) \tag{6.371}$$

with $H_m = (\omega \star \Psi_{F,m}) \in D(\mathbb{R}^n)$. Then one obtains a polynomial reproducing formula (6.361) with H_m according to (6.368) in place of $\Psi_{F,m}$.

6.5.2 Local reproducing formulas

One may ask whether one can replace the evaluation in terms of the scalar product $(f, \Psi_{F,m})$ in (6.366) by pointwise evaluation of f. This is just what one needs in the context of sampling numbers. This can be done. We formulate the outcome and indicate the connection with wavelet expansions.

Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $|\Omega| < \infty$. Let for $\tau > 0$ and $\Gamma = \partial\Omega$,

$$\Omega_\tau = \{x \in \Omega : \ \mathrm{dist}(x, \Gamma) > \tau\}. \tag{6.372}$$

Then Ω_τ is bounded. Let $C^{\mathrm{loc}}(\Omega)$ be the collection of all complex-valued continuous functions in Ω, hence

$$f \in C^{\mathrm{loc}}(\Omega) \quad \text{if, and only if,} \quad f|\Omega_\tau \in C(\Omega_\tau), \ \tau > 0, \tag{6.373}$$

in the notation of Definition 5.17. As before, $B(x, \varrho)$ stands for a ball in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ and of radius $\varrho > 0$.

Theorem 6.85. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $|\Omega| < \infty$. Let $N \in \mathbb{N}$. Then there is a number $\tau_0 > 0$ (depending on Ω), and, in dependence on N and τ_0, numbers $a > 0, b > 0, c > 0, d > 0$ with the following property. For any τ with $0 < \tau \le \tau_0$ one finds points $x^j \in \Omega_\tau$ having pairwise distance of at least $a\tau$, and real function $h_j^\tau \in D(\Omega)$ with*

$$\sup |h_j^\tau(x)| \le c, \quad \mathrm{supp}\, h_j^\tau \subset B(x^j, b\tau) \subset \Omega_{d\tau}, \tag{6.374}$$

such that the mapping U_τ,

$$U_\tau f = \sum_j f(x^j)\, h_j^\tau, \quad f \in C^{\mathrm{loc}}(\Omega), \tag{6.375}$$

is polynomial reproducing in Ω_τ,

$$(U_\tau P)(x) = P(x), \quad x \in \Omega_\tau, \tag{6.376}$$

for any polynomial of degree at most N.

Proof (Outline). Let $\{\Phi_r^j\}$ be an interior orthonormal u-wavelet basis in $L_2(\Omega)$ according to Theorem 2.33 based on Definitions 2.31 and 2.4,

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \big(f, \Phi_r^j\big)\, \Phi_r^j, \quad f \in L_2(\Omega). \tag{6.377}$$

One may choose $L \in \mathbb{N}$ in (2.32)–(2.34), based on (2.27), such that $2^{-L} \sim \tau$. Let χ_τ be a cut-off function with

$$\chi_\tau \in D\big(\Omega_{\tau/2}\big), \quad \chi_\tau(x) = 1 \text{ if } x \in \Omega_\tau. \tag{6.378}$$

Then (6.377) can be applied to $f = \chi_\tau P$, where P is a polynomial of degree at most N. We choose $N < u \in \mathbb{N}$. Then the interior wavelets in (2.33), (2.27) satisfy the moment conditions (1.88) extended to $\mathbb{R}^n$. If, in addition, $\operatorname{supp} \Phi^j_r \subset \Omega_\tau$ then

$$\big(\chi_\tau P, \Phi^j_r\big) = \big(P, \Phi^j_r\big) = 0. \tag{6.379}$$

Furthermore if $L \in \mathbb{N}$ is chosen sufficiently large in dependence on τ then one may assume that $\operatorname{supp} \Phi^j_r \subset \Omega \setminus \Omega_{\tau/2}$ for the boundary wavelets in (2.34). Hence the corresponding scalar products in (6.377) are zero. Now one obtains that

$$P(x) = \sum_{r=1}^{N_0} 2^{Ln/2} \big(\chi_\tau P, \Phi^0_r\big)\, 2^{-Ln/2}\, \Phi^0_r(x), \quad x \in \Omega_\tau. \tag{6.380}$$

By (2.31), (2.26) one has $\operatorname{supp} \Phi^0_r \subset B\big(x^{L,r}, c' 2^{-L}\big)$ taking L with $2^{-L} \sim \tau$ into account. There are points

$$\{x^{L,r,k}\}_{k=1}^{K} \subset B\big(x^{L,r}, c'' 2^{-L}\big)$$

having pairwise distance of at least $\sim 2^{-L}$, and constants $c^{L,r}_k$ with $|c^{L,r}_k| \le C$ for some $C > 0$ and all admitted L, r, k such that

$$2^{Ln/2} \int_\Omega P(y)\, \Phi^0_r(y)\, \chi_\tau(y)\, \mathrm{d}y = \sum_{k=1}^{K} c^{L,r}_k\, P\big(x^{L,r,k}\big) \tag{6.381}$$

for all polynomials of degree at most N. A detailed proof of this assertion may be found in [Tri07b], pp. 484–86. Inserting (6.381) in (6.380) one obtains that

$$P(x) = \sum_j P(x^j)\, \tilde{h}^\tau_j(x), \quad x \in \Omega_\tau, \tag{6.382}$$

where the points $x^j \in \Omega_\tau$ have the desired properties and $\tilde{h}^\tau_j$ satisfies (6.374) with $\tilde{h}$ in place of h. Finally one applies the same mollification argument as in (6.370), (6.371), assuming that ω is supported by a ball of radius $\sim \tau$. This proves (6.376) with (6.375). □

Remark 6.86. The proof of (6.381) uses in a decisive way that P is a polynomial of degree at most N, but not the specific nature of Φ^0_r (only its magnitude and support). Details may be found in [Tri07b].

6.5.3 A further comment on sampling numbers

In Definition 6.77 we introduced for the compact embeddings (6.341) in bounded Lipschitz domains Ω the (non-linear) sampling numbers $g_k(\mathrm{id})$. We obtained (6.350) for the modified embedding (6.347) with (6.349). In Section 4.3.4 we discussed to which extent one can replace bounded Lipschitz domains by more general domains in connection with entropy numbers and approximation numbers for corresponding compact embeddings. One can ask the same question for the above sampling numbers $g_k(\mathrm{id})$. This was the main aim of [Tri07b]. We formulate the outcome as an application of wavelet bases and polynomial reproducing formulas according to Theorem 6.85. In what follows the definition of the sampling numbers $g_k(\mathrm{id})$ in (6.344) is naturally extended to arbitrary compact embeddings

$$\mathrm{id}\colon G_1(\Omega) \hookrightarrow G_2(\Omega) \quad \text{with } G_1(\Omega) \hookrightarrow C(\Omega) \tag{6.383}$$

by

$$g_k(\mathrm{id}) = \inf \big[\sup \big\{ \|f\,|G_2(\Omega)\| : \ \|f\,|G_1(\Omega)\| \le 1,\ f \in G_1^\Gamma(\Omega) \big\} \big], \tag{6.384}$$

$k \in \mathbb{N}$, where

$$G_1^\Gamma(\Omega) = \big\{ f \in G_1(\Omega) : \ \textstyle\sum_{l=1}^L |f(x^l)| = 0 \big\} \quad \text{with } \Gamma = \big\{x^l\big\}_{l=1}^L \subset \Omega. \tag{6.385}$$

As before, the infimum in (6.384) is taken over all Γ with $\mathrm{card}\,\Gamma \le k$. Let σ_{pq} be as in (1.32).

Theorem 6.87. (i) *Let $u \in \mathbb{N}$ and let Ω be an arbitrary domain in $\mathbb{R}^n$ with $|\Omega| < \infty$. Let $F^{s,\mathrm{rloc}}_{pq}(\Omega)$ with*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s > \max\Big(\frac{n}{p}, \sigma_{pq}\Big) \tag{6.386}$$

($q = \infty$ if $p = \infty$) be the refined localisation spaces according to Definition 2.14. Then

$$\mathrm{id}\colon F^{s,\mathrm{rloc}}_{pq}(\Omega) \hookrightarrow L_t(\Omega), \quad 0 < t \le \infty, \tag{6.387}$$

is compact and

$$g_k(\mathrm{id}) \sim k^{-\frac{s}{n} + (\frac{1}{p} - \frac{1}{t})_+}, \quad k \in \mathbb{N}. \tag{6.388}$$

(ii) *Let $u \in \mathbb{N}$ and let Ω be an E-thick domain in $\mathbb{R}^n$ according to Definition 3.1* (ii) *with $|\Omega| < \infty$ and let $\tilde{A}^s_{pq}(\Omega)$ be the corresponding spaces as introduced in Definition 2.1* (ii) *with*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s > \frac{n}{p}, \tag{6.389}$$

($p < \infty$ for the F-spaces). Then

$$\mathrm{id}\colon \tilde{A}^s_{pq}(\Omega) \hookrightarrow L_t(\Omega), \quad 0 < t \le \infty, \tag{6.390}$$

is compact and

$$g_k(\mathrm{id}) \sim k^{-\frac{s}{p} + (\frac{1}{p} - \frac{1}{t})_+}, \quad k \in \mathbb{N}. \tag{6.391}$$

Remark 6.88. This coincides essentially with [Tri07b], Theorem 23, where one finds also a proof. It is based on the polynomial reproducing mapping U_τ in Theorem 6.85 which in turn relies on wavelet expansions. Part (ii) follows from part (i), Proposition 3.10 and some interpolation as far as the B-spaces are concerned. According to Proposition 3.8 (i) bounded Lipschitz domains are thick, and hence E-thick. This shows that (6.391) is in good agreement with (6.350). One may compare the above assertions with corresponding results in Section 4.3.4. As there

> *the adequate assumption to obtain the desired behaviour of diverse types of numbers for compact embeddings between function spaces on domains Ω is not so much that Ω is bounded, but that $|\Omega| < \infty$.*

Remark 6.89. Recall that $\mathring{A}^s_{pq}(\Omega)$ is the completion of $D(\Omega)$ in $A^s_{pq}(\mathbb{R}^n)$, Definition 5.17. If a bounded domain Ω is not only E-thick, but Lipschitz, cellular or C^∞ and if s is not an exceptional value, $s - \frac{1}{p} \notin \mathbb{N}_0$, then one can replace $\tilde{A}^s_{pq}(\Omega)$ in part (ii) of the above theorem by $\mathring{A}^s_{pq}(\Omega)$. We refer to Proposition 5.19, Remark 5.20 and Propositions 6.13 (ii), 6.15.

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